

An isoperimetric inequality in the plane with a log-convex density

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Abstract

Given a positive lower semi-continuous density f on \mathbb{R}^2 the weighted volume $V_f := f\mathcal{L}^2$ is defined on the \mathcal{L}^2 -measurable sets in \mathbb{R}^2 . The f -weighted perimeter of a set of finite perimeter E in \mathbb{R}^2 is written $P_f(E)$. We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf \left\{ P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v \right\}$$

for $v > 0$. Suppose f takes the form $f : \mathbb{R}^2 \rightarrow (0, +\infty); x \mapsto e^{h(|x|)}$ where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Our main result is the following. Let $v > 0$ and B a centred ball in \mathbb{R}^2 with $V_f(B) = v$. Then B is a minimiser for the above variational problem.

Key words: isoperimetric problem, log-convex density, generalised mean curvature

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1 Introduction

Let f be a positive lower semi-continuous density on \mathbb{R}^2 . The weighted volume $V_f := f\mathcal{L}^2$ is defined on the \mathcal{L}^2 -measurable sets in \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 . The weighted perimeter of E is defined by

$$P_f(E) := \int_{\mathbb{R}^2} f d|D\chi_E| \in [0, +\infty]. \quad (1.1)$$

We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf \left\{ P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v \right\} \quad (1.2)$$

for $v > 0$. To be more specific we suppose that f takes the form

$$f : \mathbb{R}^2 \rightarrow (0, +\infty); x \mapsto e^{h(|x|)} \quad (1.3)$$

where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Our main result is the following. It contains the classical isoperimetric inequality (cf. [8], [11]) as a special case; namely, when h is constant on $[0, +\infty)$.

Theorem 1.1. *Assume that $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function with $h \in C^1((0, +\infty))$ and let f be as in (1.3). Let $v > 0$ and B a centred ball in \mathbb{R}^2 with $V_f(B) = v$. Then B is a minimiser for (1.2).*

This result can then be strengthened using an approximation argument.

Corollary 1.2. *Assume that $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function and let f be as in (1.3). Then the conclusion of Theorem 1.1 holds.*

Corollary 1.2 is a generalisation of Conjecture 3.12 in [22] in the sense that less regularity is required of the density f : in the latter, h is supposed to be smooth on $(0, +\infty)$ as well as convex and non-decreasing. This conjecture springs in part from the observation that the weighted perimeter of a local volume-preserving perturbation of a centred ball is non-decreasing ([22] Theorem 3.10). In addition, the conjecture holds for log-convex Gaussian densities of the form $h : [0, +\infty) \rightarrow \mathbb{R}; t \mapsto e^{ct^2}$ with $c > 0$ ([3], [22] Theorem 5.2). In subsequent work partial forms of the conjecture were proved in the literature. In [17] it is shown to hold for large v provided that h is uniformly convex in the sense that $h'' \geq 1$ on $(0, +\infty)$ (see [17] Corollary 6.8). A complementary result is contained in [10] Theorem 1.1 which establishes the conjecture for small v on condition that h'' is locally uniformly bounded away from zero on $[0, +\infty)$.

The above-mentioned conjecture is proved in large part in the recent work [7] (see Theorem 1.1). There it is assumed that the function h is of class C^3 on $(0, +\infty)$ and is convex and even (meaning that h is the restriction of an even function on \mathbb{R} to $[0, +\infty)$). A uniqueness result is also obtained ([7] Theorem 1.2). We mention that Corollary 1.2 and thus Theorem 1.1 can be obtained from [7] Theorem 1.1 by a minor variation of the approximation argument used here. Our proof of Theorem 1.1 proceeds along different lines. We believe that our approach is of independent interest.

We give a brief outline of the article. In Section 2 we show that (1.2) admits an open minimiser E with C^1 boundary M (Theorem 2.9). The argument draws upon the regularity theory for almost minimal sets (cf. [25]) and includes an adaptation of [19] Proposition 3.1. In Section 3 it is shown that the boundary M is C^2 away from the origin and has constant generalised (mean) curvature (Theorem 3.5 and Theorem 3.6). We posit slightly less regularity on the density f than is assumed in the companion results in [19] and [20]. This Section also includes the result that E may be supposed to possess spherical cap symmetry (Theorem 3.10). We found the Diplomarbeit [26] and references therein helpful in preparing parts of Sections 2 and 3. In Section 4 we study the key quantity σ : the angle between a particular choice of tangent vector and the position vector measured in an anti-clockwise sense; it is defined on $M \setminus \{0\}$ and unique up to addition of integer multiples of 2π . We also introduce the set Ω : it comprises the set of all projections onto the radial coordinate of points in $M \setminus \{0\}$ at which the tangent and position vectors are not perpendicular. In Theorem 4.7 we derive a first-order ordinary differential involving the sine of σ . It is a first integral of the second-order differential equation studied in [17] (2). Sections 5 and 6 comprise an analytic interlude and are devoted to the study of solutions of the first-order differential equation that appears in Theorem 4.7 subject to Dirichlet boundary conditions. Section 6 for example contains a comparison theorem for solutions to a Riccati equation (Theorem 6.15 and Corollary 6.16). These are new as far as the author is aware. Section 7 concludes the proof of Theorem 1.1. Its crux is contained in Theorem 7.3 which asserts that the tangent at each point in $M \setminus \{0\}$ is perpendicular to the position vector so long as f is strictly increasing in the radial direction there.

We mention that the above-mentioned works as well as [22] Conjecture 3.12 are set in the context of \mathbb{R}^n for $n \geq 2$. Our proof of Theorem 1.1 does not transfer immediately to the higher-dimensional case. The main reason for this is that the generalised mean-curvature equation (4.3) acquires an extra term as in [7] Proposition 3.1. This affects the equation for y in Theorem 4.7. Apart from this, we draw on the coincidence of the essential and reduced boundaries in low dimensions in the regularity theory for almost minimal sets (cf. [25] Theorem 1.9) in Theorem 2.9; while in Theorem 4.10 we use the fact that positive curvature implies convexity for simple closed plane curves (cf. [24] Theorem 1.8).

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2 Existence and C^1 regularity

Recall that an integrable function u on \mathbb{R}^2 is said to have bounded variation if the distributional derivative of u is representable by a finite Radon measure Du (cf. [1] Definition 3.1 for example) with total variation $|Du|$. We write $u \in \text{BV}(\mathbb{R}^2)$. An \mathcal{L}^2 -measurable set E in \mathbb{R}^2 is said to have

finite perimeter if $\chi_E \in \text{BV}(\mathbb{R}^2)$. The notation \mathcal{L}^2 refers to Lebesgue measure on \mathbb{R}^2 though we also use $|\cdot|$ from time to time. As above let f be a positive lower semi-continuous density on \mathbb{R}^2 . The weighted perimeter $P_f(E)$ of E is then defined as in (1.1).

Theorem 2.1. *Assume that f is a positive radial lower-semicontinuous non-decreasing density on \mathbb{R}^2 which diverges to infinity. Then for each $v > 0$,*

- (i) *(1.2) admits a minimiser;*
- (ii) *any minimiser of (1.2) is essentially bounded.*

Proof. See [20] Theorems 3.3 and 5.9. □

Recall that a diffeomorphism $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is said to be proper if $\varphi^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact. Given $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ there exists a 1-parameter group of proper C^∞ diffeomorphisms $\varphi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as in [18] Lemma 2.99 that satisfy

$$\begin{aligned} \partial_t \varphi(t, x) &= X(\varphi(t, x)) \text{ for each } (t, x) \in \mathbb{R} \times \mathbb{R}^2; \\ \varphi(0, x) &= x \text{ for each } x \in \mathbb{R}^2. \end{aligned} \tag{2.1}$$

Let E be a set of finite perimeter in \mathbb{R}^2 . The first variation of weighted volume resp. perimeter along $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is defined by

$$\delta V_f(X) := \left. \frac{d}{dt} \right|_{t=0} V_f(\varphi_t(E)), \tag{2.2}$$

$$\delta P_f(X) := \left. \frac{d}{dt} \right|_{t=0} P_f(\varphi_t(E)), \tag{2.3}$$

whenever the limit exists.

Proposition 2.2. *Let f be a positive lower-semicontinuous density on \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 and $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then*

$$\delta V_f(X) = - \int_{\mathcal{F}E} f \langle \nu^E, X \rangle d\mathcal{H}^1.$$

We use \mathcal{H}^k ($k \in [0, +\infty)$) to stand for k -dimensional Hausdorff measure. The reduced boundary $\mathcal{F}E$ of E is defined by

$$\mathcal{F}E := \left\{ x \in \text{supp}|D\chi_E| : \nu_E(x) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B(x, \rho))}{|D\chi_E|(B(x, \rho))} \text{ exists in } \mathbb{R}^2 \text{ and } |\nu_E(x)| = 1 \right\}$$

(cf. [1] Definition 3.54) and is a Borel set (cf. [1] Theorem 2.22 for example). As usual $B(x, \rho)$ denotes the open ball in \mathbb{R}^2 with centre $x \in \mathbb{R}^2$ and radius $\rho > 0$.

Proof. By the area formula ([1] Theorem 2.71 and (2.74)),

$$V_f(\varphi_t(E)) = \int_{\varphi_t(E)} f dx = \int_E (f \circ \varphi_t) J_2 d(\varphi_t)_x dx. \tag{2.4}$$

On account of Taylor's Theorem we may write $\varphi(t, x) = x + tX(x) + o(t)$. Moreover, $d\varphi(t, \cdot) = I + t dX + o(t)$. The representing matrix for $d\varphi(t, \cdot)$ with respect to the standard orthonormal basis $\{e_1, e_2\}$ for \mathbb{R}^2 has entries

$$a_{jk} = \delta_{jk} + t \langle dX e_j, e_k \rangle + o(t).$$

By [1] Definition 2.68 and Proposition 2.69, $J_2 d\varphi_t = 1 + t \text{div } X + o(t)$. Therefore

$$\begin{aligned} V_f(\varphi_t(E)) - V_f(E) &= \int_E (f \circ \varphi_t) J_2 d\varphi_t - \int_E f dx \\ &= \int_E (f \circ \varphi_t) (J_2 d\varphi_t - 1) dx + \int_E f \circ \varphi_t - f dx \\ &= t \int_E f \text{div } X dx + t \int_E \langle \nabla f, X \rangle dx + o(t) \end{aligned}$$

and hence

$$\delta V_f(X) = \int_E \operatorname{div}(fX) dx = - \int_{\mathcal{F}E} f \langle \nu^E, X \rangle d\mathcal{H}^1$$

by the generalised Gauss-Green formula [1] Theorem 3.36. \square

Let E be a \mathcal{L}^2 -measurable set in \mathbb{R}^2 . The set of points in E with density $t \in [0, 1]$ is given by

$$E^t := \left\{ x \in \mathbb{R}^2 : \lim_{\rho \downarrow 0} \frac{|E \cap B(x, \rho)|}{|B(x, \rho)|} = t \right\}.$$

The set E^1 is the measure-theoretic interior of E while E^0 is the measure-theoretic exterior of E . The essential boundary of E is the set $\partial^* E := \mathbb{R}^2 \setminus (E^0 \cup E^1)$. If E is a set of finite perimeter in \mathbb{R}^2 then

$$\mathcal{F}E \subset E^{1/2} \subset \partial^* E \text{ and } \mathcal{H}^1(\partial^* E \setminus \mathcal{F}E) = 0 \quad (2.5)$$

by [1] Theorem 3.61.

Lemma 2.3. *Let φ be a proper C^1 diffeomorphism of \mathbb{R}^2 and E a set of finite perimeter in \mathbb{R}^2 . Then*

- (i) $\varphi(E)$ is a set of finite perimeter in \mathbb{R}^2 ;
- (ii) $\partial^* \varphi(E) = \varphi(\partial^* E)$;
- (iii) $\mathcal{F} \varphi(E) \equiv \varphi(\mathcal{F}E)$ modulo \mathcal{H}^1 .

Proof. Part (i) follows from [1] Theorem 3.16. Given $x \in E^0$ we claim that $y := \varphi(x) \in \varphi(E)^0$. Let V be a relatively compact open set in \mathbb{R}^2 containing y . Let L stand for the Lipschitz constant of φ^{-1} restricted to V . Also let U be a relatively compact open set in \mathbb{R}^2 containing x and let M stand for the Lipschitz constant of φ restricted to U . Note that $B(y, r) \subset \varphi(B(x, Lr))$ for $r > 0$ small. By [1] Theorem 2.53, the fact that φ is a bijection, and [1] Proposition 2.49,

$$\begin{aligned} |\varphi(E) \cap B(y, r)| &= \mathcal{L}^2(\varphi(E) \cap B(y, r)) \\ &\leq \mathcal{L}^2(\varphi(E) \cap \varphi(B(x, Lr))) \\ &= \mathcal{L}^2(\varphi(E \cap B(x, Lr))) \\ &\leq M^2 \mathcal{L}^2(E \cap B(x, Lr)) = M^2 |E \cap B(x, Lr)|. \end{aligned}$$

This means that

$$\frac{|\varphi(E) \cap B(y, r)|}{|B(y, r)|} \leq (LM)^2 \frac{|E \cap B(x, Lr)|}{|B(x, Lr)|}$$

for $r > 0$ small and this proves the claim. This entails that $\varphi(E^0) \subset [\varphi(E)]^0$. The reverse inclusion can be seen using the fact that φ is a bijection. In summary $\varphi(E^0) = [\varphi(E)]^0$. The observation that $E^1 = (\mathbb{R}^2 \setminus E)^0$ can be used to establish the remainder of (ii). Item (iii) follows from (2.5) and (ii). \square

Proposition 2.4. *Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let E be a relatively compact set in \mathbb{R}^2 with finite perimeter. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then*

$$\delta P_f(X) = \int_{\mathcal{F}E} \langle \nabla f, X \rangle + f \operatorname{div}^{\mathcal{F}E} X d\mathcal{H}^1.$$

The tangential divergence $\operatorname{div}^{\mathcal{F}E} X$ is defined as in [1] Definition 7.27.

Proof. First note that

$$P_f(\varphi_t(E)) = \int_{\mathcal{F}\varphi_t(E)} f d\mathcal{H}^1 = \int_{\varphi_t(\mathcal{F}E)} f d\mathcal{H}^1$$

by Lemma 2.3. As $\mathcal{F}E$ is countably 1-rectifiable ([1] Theorem 3.59) we may use the generalised area formula [1] Theorem 2.91 to write

$$P_f(\varphi_t(E)) = \int_{\mathcal{F}E} (f \circ \varphi_t) J_1 d^{\mathcal{F}E}(\varphi_t)_x d\mathcal{H}^1.$$

As in the proof of [1] Theorem 7.31,

$$J_1 d^{\mathcal{F}E}(\varphi_t)_x = 1 + t \operatorname{div}^{\mathcal{F}E} X + o(t).$$

The reduced boundary $\mathcal{F}E$ of E is relatively compact in \mathbb{R}^2 . For each $x \in \mathcal{F}E$ and any $t \in \mathbb{R}$,

$$|(f \circ \varphi_t)(x) - f(x)| \leq K|\varphi(t, x) - x| \leq K\|X\|_\infty t$$

where K is the Lipschitz constant of f on a relatively compact open neighbourhood of $\mathcal{F}E$. Moreover,

$$\lim_{t \rightarrow 0} (1/t)[(f \circ \varphi_t) - f] = \langle \nabla f, X \rangle \mathcal{H}^1\text{-a.e. on } \mathcal{F}E$$

by Rademacher's Theorem (cf. [1] Theorem 2.14) and the fact that $\mathcal{F}E$ is countably 1-rectifiable (cf. [1] Theorem 3.59). The claim follows on taking the limit $t \rightarrow 0$ in

$$(1/t)[P_f(\varphi_t(E)) - P_f(E)] = (1/t) \int_{\mathcal{F}E} (f \circ \varphi_t)(J_1 d^{\mathcal{F}E}(\varphi_t)_x - 1) + [f \circ \varphi_t - f] d\mathcal{H}^1$$

by appeal to the dominated convergence theorem. \square

Lemma 2.5. *Let f be a positive lower semi-continuous density on \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 and $p \in \mathcal{F}E$. For any $r > 0$ there exists $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[X] \subset B(p, r)$ such that*

$$\delta V_f(X) = 1 \text{ and } |\delta P_f(X)| < \infty.$$

Proof. Note that

$$P_f(E, B(p, r)) = \int_{B(p, r)} f d|D\chi_E| = \int_{B(p, r) \cap \mathcal{F}E} f d\mathcal{H}^1 > 0$$

for any $r > 0$ by [1] Theorem 3.59 and (3.57) in particular. By the variational characterisation of the f -perimeter relative to $B(p, r)$ we can find $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[Y] \subset B(p, r)$ such that

$$0 < \int_{E \cap B(p, r)} \operatorname{div}(fY) dx = - \int_{\mathcal{F}E \cap B(p, r)} f \langle \nu^E, Y \rangle d\mathcal{H}^1 =: c$$

where we make use of the generalised Gauss-Green formula (cf. [1] Theorem 3.36). Put $X := (1/c)Y$. Then $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[X] \subset B(p, r)$ and $\delta V_f(X) = 1$ according to Proposition 2.2. By Proposition 2.4, $|\delta P_f(X)| < \infty$. \square

Proposition 2.6. *Let f be a positive lower semi-continuous density on \mathbb{R}^2 . Let U be an open set in \mathbb{R}^2 with bounded Lipschitz boundary. Let E, F_1, F_2 be sets of finite perimeter in \mathbb{R}^2 . Assume that $E \Delta F_1 \subset \subset U$, $E \Delta F_2 \subset \subset \mathbb{R}^2 \setminus \overline{U}$. Define*

$$F := [F_1 \cap U] \cup [F_2 \setminus U].$$

Then F is a set of finite perimeter in \mathbb{R}^2 and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

Proof. The function $\chi_E|_U \in \text{BV}(U)$. Moreover, $D(\chi_E|_U) = (D\chi_E)|_U$. We write χ_E^U for the boundary trace of $\chi_E|_U$ (see [1] Theorem 3.87); then $\chi_E^U \in L^1(\partial U, \mathcal{H}^1 \llcorner \partial U)$ (cf. [1] Theorem 3.88). We use similar notation elsewhere. By [1] Corollary 3.89,

$$\begin{aligned} D\chi_E &= D\chi_E \llcorner U + (\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}) \nu_U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ D\chi_F &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \overline{U}}) \nu_U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ D\chi_{F_1} &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}) \nu_U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ D\chi_{F_2} &= D\chi_E \llcorner U + (\chi_E^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \overline{U}}) \nu_U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \overline{U}). \end{aligned}$$

From the definition of the total variation measure ([1] Definition 1.4),

$$\begin{aligned} |D\chi_E| &= |D\chi_E| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}| \mathcal{H}^1 \llcorner \partial U + |D\chi_E| \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ |D\chi_F| &= |D\chi_{F_1}| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}| \mathcal{H}^1 \llcorner \partial U + |D\chi_{F_2}| \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ |D\chi_{F_1}| &= |D\chi_{F_1}| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}| \mathcal{H}^1 \llcorner \partial U + |D\chi_E| \llcorner (\mathbb{R}^2 \setminus \overline{U}); \\ |D\chi_{F_2}| &= |D\chi_E| \llcorner U + |\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \overline{U}}| \mathcal{H}^1 \llcorner \partial U + |D\chi_{F_2}| \llcorner (\mathbb{R}^2 \setminus \overline{U}); \end{aligned}$$

where we also use the fact that $\chi_E^U = \chi_{F_1}^U$ as $E \Delta F_1 \subset\subset U$ and similarly for $\mathbb{R}^2 \setminus \overline{U}$. The result now follows. \square

Proposition 2.7. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Let $v > 0$ and E be a minimiser of (1.2). Assume that E is bounded. There exist constants $C > 0$ and $\delta > 0$ with the following property. For any $x \in \mathbb{R}^2$ and $0 < r < \delta$,*

$$P_f(E) - P_f(F) \leq C |V_f(E) - V_f(F)| \quad (2.6)$$

where F is any set with finite perimeter in \mathbb{R}^2 such that $E \Delta F \subset\subset B(x, r)$.

Proof. The proof follows the outline of [19] Proposition 3.1. We assume to the contrary that

$$(\forall C > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}^2)(\exists r \in (0, \delta))(\exists F \subset \mathbb{R}^2)$$

$$\left[F \Delta E \subset\subset B(x, r) \wedge \Delta P_f > C |\Delta V_f| \right] \quad (2.7)$$

in the language of quantifiers where we have taken some liberties with notation.

Choose $p_1, p_2 \in \mathcal{F}E$ with $p_1 \neq p_2$. Choose $r_0 > 0$ such that the open balls $B(p_1, r_0)$ and $B(p_2, r_0)$ are disjoint. Choose vector fields $X_j \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X_j] \subset B(p_j, r_0)$ such that

$$\delta V_f(X_j) = 1 \text{ and } |\delta P_f(X_j)| < \infty \text{ for } j = 1, 2 \quad (2.8)$$

as in Lemma 2.5. Put $a := \max\{\delta P_f(X_1), \delta P_f(X_2)\}$. Let $\varphi^{(j)}$ stand for the 1-parameter group of C^∞ diffeomorphisms of \mathbb{R}^2 associated to the vector fields X_j for $j = 1, 2$ as in (2.1). By (2.8),

$$V_f(\varphi_t^{(j)}(E)) - V_f(E) = t + o(t) \text{ as } t \rightarrow 0 \text{ for } j = 1, 2.$$

So there exist $\varepsilon > 0$ and $1/2 > \eta > 0$ such that

$$t - \eta|t| < V_f(\varphi_t^{(j)}(E)) - V_f(E) < t + \eta|t|; \quad (2.9)$$

$$|P_f(\varphi_t^{(j)}(E)) - P_f(E)| < 2a|t|;$$

for $|t| < \varepsilon$ and $j = 1, 2$. In particular,

$$\begin{aligned} |V_f(\varphi_t^{(j)}(E)) - V_f(E)| &> (1 - \eta)|t|; \\ |P_f(\varphi_t^{(j)}(E)) - P_f(E)| &< 4a|V_f(\varphi_t^{(j)}(E)) - V_f(E)| \text{ for } |t| < \varepsilon; \end{aligned} \quad (2.10)$$

for $|t| < \varepsilon$ and $j = 1, 2$.

In (2.7) choose $C = 4a$ and $\delta > 0$ such that

- (a) $0 < 2\delta < \text{dist}(B(p_1, r_0), B(p_2, r_0))$,
- (b) $\sup\{V_f(B(x, \delta)) : x \in I_{2\delta}(E)\} < (1 - \eta)\varepsilon$.

Choose x, r and F_1 as in (2.7). In light of (a) we may assume that $B(x, r) \cap B(p_1, r_0) = \emptyset$. By (b),

$$|V_f(F_1) - V_f(E)| \leq V_f(B(x, r)) \leq V_f(B(x, \delta)) < (1 - \eta)\varepsilon. \quad (2.11)$$

From (2.9) and (2.11) we can find $t \in (-\varepsilon, \varepsilon)$ such that with $F_2 := \varphi_t^{(1)}(E)$,

$$V_f(F_2) - V_f(E) = -\{V_f(F_1) - V_f(E)\} \quad (2.12)$$

by the intermediate value theorem. From (2.7),

$$P_f(F_1) < P_f(E) - C|V_f(F_1) - V_f(E)| \quad (2.13)$$

while from (2.10),

$$P_f(F_2) < P_f(E) + C|V_f(F_2) - V_f(E)|. \quad (2.14)$$

Let F be the set

$$F := [F_1 \setminus B(p_1, r_0)] \cup [B(p_1, r_0) \cap F_2].$$

Note that $E \Delta F_2 \subset \subset B(p_1, r_0)$. By Proposition 2.6 F is a set of finite perimeter in \mathbb{R}^2 and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

We then infer from (2.13), (2.14) and (2.12) that

$$\begin{aligned} P_f(F) &= P_f(F_1) + P_f(F_2) - P_f(E) \\ &< P_f(E) - C|V_f(F_1) - V_f(E)| + P_f(E) + C|V_f(F_2) - V_f(E)| - P_f(E) = P_f(E). \end{aligned}$$

On the other hand, $V_f(F) = V_f(F_1) + V_f(F_2) - V_f(E) = V_f(E)$ by (2.12). We therefore obtain a contradiction to the f -isoperimetric property of E . \square

Let E be a set of finite perimeter in \mathbb{R}^2 and A a relatively compact open set in \mathbb{R}^2 . The minimality excess is the function ψ defined by

$$\begin{aligned} \nu(E, A) &:= \inf\{P(F, A) : \chi_F \in \text{BV}(\mathbb{R}^2) \text{ and } F \Delta E \subset \subset A\}; \\ \psi(E, A) &:= P(E, A) - \nu(E, A); \end{aligned} \quad (2.15)$$

as in [25] (1.9). We recall that the boundary of E is said to be almost minimal in \mathbb{R}^2 if for each open relatively compact set U in \mathbb{R}^2 there exists $T > 0$ and a positive constant K such that for every $x \in U$ and $r \in (0, T)$,

$$\psi(E, B(x, r)) \leq Kr^2. \quad (2.16)$$

This definition corresponds to [25] Definition 1.5 with a particular choice of α .

Theorem 2.8. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Let E be a minimiser of (1.2). Then the boundary of E is almost minimal in \mathbb{R}^2 .*

Proof. Let U be a relatively compact open set in \mathbb{R}^2 and $\delta > 0$ as in Proposition 2.7. The open δ -neighbourhood of U is denoted $I_\delta(U)$. Let $x \in U$ and $r \in (0, \delta)$. Put $V := I_{2\delta}(U)$. For the sake

of brevity write $m := \inf_{B(x,r)} f$ and $M := \sup_{B(x,r)} f$. Let F be a set of finite perimeter in \mathbb{R}^2 such that $F \Delta E \subset\subset B(x, r)$. By Proposition 2.7,

$$\begin{aligned}
& P(E, B(x, r)) - P(F, B(x, r)) \\
& \leq \frac{1}{m} P_f(E, B(x, r)) - \frac{1}{M} P_f(F, B(x, r)) \\
& = \frac{1}{m} \left(P_f(E, B(x, r)) - P_f(F, B(x, r)) \right) + \left(\frac{1}{m} - \frac{1}{M} \right) P_f(F, B(x, r)) \\
& \leq \frac{1}{m} \left(P_f(E, B(x, r)) - P_f(F, B(x, r)) \right) + \frac{M-m}{m^2} P_f(F, B(x, r)) \\
& \leq \frac{C}{\inf_V f} |V_f(E) - V_f(F)| + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} P(F, B(x, r)) \\
& \leq \frac{C\pi}{\inf_V f} r^2 + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} P(F, B(x, r))
\end{aligned}$$

where L stands for the Lipschitz constant of the restriction of f to V . We then derive that

$$\psi(E, B(x, r)) \leq \frac{C\pi}{\inf_V f} r^2 + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} \nu(E, B(x, r)).$$

By [12] (5.14), $\nu(E, B(x, r)) \leq \pi r$. The inequality in (2.16) now follows. \square

Theorem 2.9. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Let $v > 0$ and E a minimiser of (1.2). Then E^1 is a bounded open set in \mathbb{R}^2 , $V_f(E^1) = v$, $P_f(E^1) = I_f(v)$ and $\partial E^1 = \partial^* E$ is a C^1 hypersurface in \mathbb{R}^2 .*

Proof. By [25] Theorem 1.9 (see also [12] Chapter 4), $\partial^* E$ is a C^1 hypersurface in \mathbb{R}^2 (taking note of differences in notation). It follows that E^1 is open. The latter set is equivalent to E in the sense of Lebesgue measure (cf. [12] Proposition 3.1) and so is bounded with identical f -measure and f -perimeter as E . \square

3 Higher regularity and spherical cap symmetry

Let I be an open interval in \mathbb{R} containing 0. Let $Z : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; (t, x) \mapsto Z(t, x)$ be a continuous time-dependent vector field on \mathbb{R}^2 with the properties

$$(Z.1) \quad Z(t, \cdot) \in C_c^1(\mathbb{R}^2, \mathbb{R}^2) \text{ for each } t \in I;$$

$$(Z.2) \quad \bigcup_{t \in I} \text{supp}[Z(t, \cdot)] \subset \mathbb{R}^2 \text{ is relatively compact.}$$

By [15] Theorems I.1.1, I.2.1, I.3.1, I.3.3 there exists a unique flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$(F.1) \quad \varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is of class } C^1;$$

$$(F.2) \quad \varphi(0, x) = x \text{ for each } x \in \mathbb{R}^2;$$

$$(F.3) \quad \partial_t \varphi(t, x) = Z(t, \varphi(t, x)) \text{ for each } (t, x) \in I \times \mathbb{R}^2;$$

$$(F.4) \quad \varphi_t := \varphi(t, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is a proper diffeomorphism for each } t \in I.$$

Given a set E of finite perimeter in \mathbb{R}^2 the first variation $\delta V_f(Z)$ resp. $\delta P_f(Z)$ of weighted volume and perimeter along Z are defined as in (2.2) and (2.3).

Suppose the open set E in \mathbb{R}^2 has C^1 boundary M . Let $p \in M$ and e_1 a unit tangent vector to M at p . There exists a local parametrisation $\gamma_1 : I \rightarrow M$ where $I = (-\delta, \delta)$ for some $\delta > 0$ of class C^1 with $\gamma_1(0) = p$. We may assume that γ_1 is parametrised by arc-length and that $\gamma_1'(0) = e_1$. Let X be a vector field defined in some neighbourhood of p in M . Then

$$(D_{e_1} X)(p) := \left. \frac{d}{ds} \right|_{s=0} (X \circ \gamma_1)(s) \tag{3.1}$$

if this limit exists and the divergence $\operatorname{div}^M X$ of X along M at p is defined by

$$\operatorname{div}^M X := \langle D_{e_1} X, e_1 \rangle \quad (3.2)$$

evaluated at p . Suppose that X is a vector field in $C^1(U, \mathbb{R}^2)$ where U is an open neighbourhood of p in \mathbb{R}^2 . Choose a unit normal vector n at p such that the pair $\{e_1, n\}$ form a positively oriented orthonormal basis for \mathbb{R}^2 . Then

$$\operatorname{div} X = \operatorname{div}^M X + \langle D_n X, n \rangle \quad (3.3)$$

at p .

Proposition 3.1. *Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let E be a bounded open set in \mathbb{R}^2 with C^1 boundary M and finite perimeter $\mathcal{H}^1(M) < +\infty$. Let Z be a time-dependent vector field as above. Put $Z_0 := Z(0, \cdot) \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$. Then*

$$\delta P_f(Z) = \int_M \langle \nabla f, Z_0 \rangle + f \operatorname{div}^M Z_0 \, d\mathcal{H}^1.$$

Proof. We first remark that the flow $\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to Z is continuously differentiable in t, x in virtue of (Z.1) by [15] Theorem I.3.3. Put $y(t, x) := d\varphi(t, x) = d\varphi_t(x)$ for $(t, x) \in I \times \mathbb{R}^2$. By [15] Theorem I.3.3,

$$\dot{y}(t, x) = dZ(t, \varphi(t, x))y(t, x)$$

for each $(t, x) \in I \times \mathbb{R}^2$ and $y(0, x) = I_2$ for each $x \in \mathbb{R}^2$ where I_2 stands for the 2×2 -identity matrix.

As in Proposition 2.4,

$$P_f(\varphi_t(E)) = \int_{\varphi_t(M)} f \, d\mathcal{H}^1$$

with the help of Lemma 2.3. By the generalised area formula [1] Theorem 2.91 we may write

$$P_f(\varphi_t(E)) = \int_M (f \circ \varphi_t) J_1 d^M(\varphi_t)_x \, d\mathcal{H}^1.$$

Let $x \in M$ and τ_1 a unit vector in the tangent space $\operatorname{Tan}^1(M, x)$ to M at x . As in the proof of [1] Theorem 7.31,

$$J_1 d^M(\varphi_t)_x = \left\{ \langle d(\varphi_t)_x \tau_1, e_1 \rangle^2 + \langle d(\varphi_t)_x \tau_1, e_2 \rangle^2 \right\}^{1/2}$$

where $\{e_1, e_2\}$ stands for the standard basis for \mathbb{R}^2 . For $x \in \mathbb{R}^2$ and $t \in I$,

$$\begin{aligned} d(\varphi_t)_x &= d\varphi(t, x) = I_2 + d\varphi(t, x) - d\varphi(0, x) \\ &= I_2 + t\dot{y}(0, x) + t \left\{ \frac{d\varphi(t, x) - d\varphi(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I_2 + tdZ(0, x) + t \left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I_2 + tdZ_0(x) + t \left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\}. \end{aligned}$$

Applying the mean-value theorem component-wise and using continuity of the matrix \dot{y} in its arguments we see that

$$\frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \rightarrow 0 \text{ as } t \rightarrow 0$$

locally uniformly on \mathbb{R}^2 . In particular,

$$J_1 d^M(\varphi_t)_x = \left\{ 1 + 2t \langle dZ_0 \tau_1, \tau_1 \rangle + o(t) \right\}^{1/2} = 1 + t \operatorname{div}^M(Z_0) + o(t)$$

where the term $o(t)$ has the property that $t^{-1}o(t) \rightarrow 0$ uniformly on M as $I \ni t \rightarrow 0$. The remainder of the proof follows as in Proposition 2.4. \square

Given $X, Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of \mathbb{R}^2 associated to the vector fields X resp. Y as in (2.1). Let I be an open interval in \mathbb{R} containing the point 0. Suppose that the function $\sigma : I \rightarrow \mathbb{R}$ is C^1 with $\sigma(0) = 0$ and $\sigma'(0) = 1$. Define a flow via

$$\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; t \mapsto \chi(\sigma(t), \psi(t, x)).$$

Lemma 3.2. *The time-dependent vector field Z associated with the isotopy φ is given by*

$$Z(t, x) = \sigma'(t)Y(\chi(\sigma(t), \psi(t, x))) + d\chi(\sigma(t), \psi(t, x))X(\psi(t, x)) \quad (3.4)$$

and satisfies (Z.1) and (Z.2).

Proof. For $t \in I$ and $x \in \mathbb{R}^2$ we compute using (2.1),

$$\begin{aligned} \partial_t \varphi(t, x) &= (\partial_t \chi)(\sigma(t), \psi(t, x))\sigma'(t) + d\chi(\sigma(t), \psi(t, x))\partial_t \psi(t, x) \\ &= \sigma'(t)Y(\chi(\sigma(t), \psi(t, x))) + d\chi(\sigma(t), \psi(t, x))X(\psi(t, x)). \end{aligned}$$

\square

Lemma 3.3. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Let $v > 0$ and E a bounded minimiser of (1.2). Then there exists $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with the property*

$$\int_E \operatorname{div}(fY) dx = 1.$$

Proof. An argument using comparison with the classical isoperimetric inequality shows that $I_f(v) > 0$. By [1] Proposition 1.23 and Proposition 1.47,

$$\begin{aligned} 0 < P_f(E) &= \sup \left\{ \int_E \operatorname{div}(fX) dx : X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2), \|X\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_E \operatorname{div}(fX) dx : X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2), \|X\|_\infty \leq 1 \right\} \end{aligned}$$

so that we may choose $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\int_E \operatorname{div}(fX) dx > 0$. The assertion follows by scaling. \square

Proposition 3.4. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Given $v > 0$ let E be a minimiser of (1.2). Assume that E is a bounded open set in \mathbb{R}^2 with C^1 boundary M . Then there exists $\lambda \in \mathbb{R}$ such that for any $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$,*

$$\int_M \left\{ \langle \nabla f, X \rangle + f \operatorname{div}^M X + \lambda f \langle n, X \rangle \right\} d\mathcal{H}^1 = 0.$$

Proof. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Choose $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ as in Lemma 3.3. Let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of \mathbb{R}^2 associated to the vector fields X resp. Y

as in (2.1). For each $(s, t) \in \mathbb{R}^2$ the set $\chi_s(\psi_t(E))$ is an open set in \mathbb{R}^2 with C^1 boundary and $\partial(\chi_s \circ \psi_t)(E) = (\chi_s \circ \psi_t)(M)$ by Lemma 2.3. Define

$$\begin{aligned} V(s, t) &:= V_f(\chi_t(\psi_s(E))) - V_f(E), \\ P(s, t) &:= P_f(\chi_t(\psi_s(E))), \end{aligned}$$

for $(s, t) \in \mathbb{R}^2$. We write $F = (\chi_t \circ \psi_s)(E)$. Arguing as in Proposition 2.4,

$$\begin{aligned} \partial_t V(s, t) &= \lim_{h \rightarrow 0} (1/h) \{V_f(\chi_h(F)) - V_f(F)\} = \int_F \operatorname{div}(fY) \, dx \\ &= \int_E (\operatorname{div}(fY) \circ \chi_t \circ \psi_s) J_2 d(\chi_t \circ \psi_s)_x \, dx \end{aligned}$$

with an application of the area formula (cf. [1] Theorem 2.71). This last varies continuously in (s, t) . The same holds for partial differentiation with respect to s . Indeed, put $\eta := \chi_t \circ \psi_s$. Then noting that $J_2 d(\eta \circ \psi_h) = (J_2 d\eta) \circ \psi_h \cdot J_2 d\psi_h$ and using the dominated convergence theorem,

$$\begin{aligned} \partial_s V(s, t) &= \lim_{h \rightarrow 0} (1/h) \{V_f(\eta(\psi_h(E))) - V_f(\eta(E))\} \\ &= \lim_{h \rightarrow 0} (1/h) \left\{ \int_E (f \circ \eta \circ \psi_h) J_2 d(\eta \circ \psi_h)_x \, dx - \int_E (f \circ \eta) J_2 d\eta_x \, dx \right\} \\ &= \lim_{h \rightarrow 0} (1/h) \left\{ \int_E [(f \circ \eta \circ \psi_h) - (f \circ \eta)] J_2 d(\eta \circ \psi_h)_x \, dx \right. \\ &\quad \left. + \int_E (f \circ \eta) [(J_2 d\eta \circ \psi_h - J_2 d\eta) J_2 d\psi_h] \, dx + \int_E (f \circ \eta) J_2 d\eta [J_2 d\psi_h - 1] \, dx \right\} \\ &= \int_E \langle \nabla(f \circ \eta), X \rangle J_2 d\eta_x \, dx + \int_E (f \circ \eta) \langle \nabla J_2 d\eta, X \rangle \, dx + \int_E (f \circ \eta) J_2 d\eta \operatorname{div} X \, dx \end{aligned}$$

where the explanation for the last term can be found in the proof of Proposition 2.2. In this regard we note that $d(\chi_t)$ (for example) is continuous on $I \times \mathbb{R}^2$ (cf. [1] Theorem 3.3 and Exercise 3.2) and in particular $\nabla J_2 d\chi_t$ is continuous on $I \times \mathbb{R}^2$. The expression above also varies continuously in (s, t) as can be seen with the help of the dominated convergence theorem. This means that $V(\cdot, \cdot)$ is continuously differentiable on \mathbb{R}^2 . Note that

$$\partial_t V(0, 0) = \int_E \operatorname{div}(fY) \, dx = 1$$

by choice of Y . By the implicit function theorem there exists $\eta > 0$ and a C^1 function $\sigma : (-\eta, \eta) \rightarrow \mathbb{R}$ such that $\sigma(0) = 0$ and $V(s, \sigma(s)) = 0$ for $s \in (-\eta, \eta)$; moreover,

$$\sigma'(0) = -\partial_s V(0, 0) = - \int_E \left\{ \langle \nabla f, X \rangle + f \operatorname{div} X \right\} \, dx = - \int_E \operatorname{div}(fX) \, dx = \int_M f \langle n, X \rangle \, d\mathcal{H}^1$$

by the Gauss-Green formula (cf. [1] Theorem 3.36).

The mapping

$$\varphi : (-\eta, \eta) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; t \mapsto \chi(\sigma(t), \psi(t, x))$$

satisfies conditions (F.1)-(F.4) above with $I = (-\eta, \eta)$ where the associated time-dependent vector field Z is given as in (3.4) and satisfies (Z.1) and (Z.2). The mapping $I \rightarrow \mathbb{R}; t \mapsto P_f(\varphi_t(E))$ is C^1 as can be seen from Proposition 3.1 and has a local minimum at $t = 0$. By Proposition 3.1 and Lemma 3.2,

$$\begin{aligned} 0 &= \delta P_f(Z) = \int_M \langle \nabla f, Z_0 \rangle + f \operatorname{div}^M Z_0 \, d\mathcal{H}^1 \\ &= \int_M \langle \nabla f, X \rangle + f \operatorname{div}^M X \, d\mathcal{H}^1 + \sigma'(0) \int_M \langle \nabla f, Y \rangle + f \operatorname{div}^M Y \, d\mathcal{H}^1 \end{aligned}$$

noting that $Z_0 = \sigma'(0)Y + X$. The identity then follows with $\lambda = \int_M \langle \nabla f, Y \rangle + f \operatorname{div}^M Y \, d\mathcal{H}^1$. By a density argument the claim follows for $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$. \square

We compare the following Theorem with [19] 3.10 and [20] Theorem 2.1. The density f may have a singularity at the origin though is locally Lipschitz on \mathbb{R}^2 .

Theorem 3.5. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Given $v > 0$ there exists a minimiser E of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 and $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 .*

We write f in the form

$$f(x) = \mathbf{f}(|x|) = e^{h(|x|)} \text{ for each } x \in \mathbb{R}^2$$

where $h : [0, +\infty) \rightarrow \mathbb{R}$ is locally Lipschitz continuous, C^1 on $(0, +\infty)$ and diverges to infinity. We write $h' = \varrho$ on $(0, +\infty)$.

Proof. By Theorem 2.9 we may assume that E is open and that M is a C^1 hypersurface in \mathbb{R}^2 . Let $p \in M \setminus \{0\}$. By rotating axes if necessary we may assume that there exists an open interval I in $\mathbb{R} \setminus \{0\}$, $R > 0$ and $u \in C^1(I)$ such that

$$E \cap Q = \{(x, y) \in Q : x \in I \text{ and } y < u(x)\}$$

where $Q := I \times (-R, R)$ and $p \in M \cap Q$. Define

$$\varphi : I \rightarrow M \cap Q; x \mapsto (x, u(x)).$$

Let $\zeta \in C_c^1(I)$ and $\phi \in C_c^1(\mathbb{R})$ be equal to unity on a neighbourhood of $(-R, R)$ and define $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ by

$$X(x, y) := \zeta(x)\phi(y)e_2 \text{ for } (x, y) \in \mathbb{R}^2.$$

The inner unit normal on $M \cap Q$ is parametrised by

$$n \circ \varphi = \frac{(-u', 1)}{\sqrt{1 + (u')^2}}$$

on I . On the other hand,

$$\operatorname{div}^M X = \operatorname{div} X - \langle n, D_n X \rangle = -\langle n, D_n X \rangle = -\langle n, dXn \rangle \text{ on } M \cap Q$$

as in (3.3) and hence

$$(\operatorname{div}^M X) \circ \varphi = \frac{u' \zeta'}{1 + (u')^2} \text{ on } I.$$

The Jacobean determinant of the parametrisation φ is $J_1 d\varphi = \sqrt{1 + (u')^2}$. From Proposition 3.4 and the area formula [1] Theorem 2.71,

$$\begin{aligned} 0 &= \int_M \left\{ \langle \nabla f, X \rangle + f \operatorname{div}^M X + \lambda f \langle n, X \rangle \right\} d\mathcal{H}^1 \\ &= \int_I \left\{ \langle \nabla f \circ \varphi, X \circ \varphi \rangle + (f \circ \varphi) \operatorname{div}^M X \circ \varphi + \lambda f \circ \varphi \langle n \circ \varphi, X \circ \varphi \rangle \right\} J_1 d\varphi \, dx \\ &= \int_I \left[a(x, u) \sqrt{1 + (u')^2} + \lambda b(x, u) \right] \zeta + b(x, u) \frac{u'}{\sqrt{1 + (u')^2}} \zeta' \, dx \end{aligned} \quad (3.5)$$

where the functions $a, b : I \times \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$\begin{aligned} a(x, z) &:= z \frac{(\mathbf{f} \varrho)(\sqrt{x^2 + z^2})}{\sqrt{x^2 + z^2}}; \\ b(x, z) &:= \mathbf{f}(\sqrt{x^2 + z^2}). \end{aligned}$$

Define $j : I \times \mathbb{R} \times \mathbb{R}$ by

$$j(x, z, p) := \mathbf{f}(\sqrt{x^2 + z^2})\sqrt{1 + p^2} + \lambda c(x, z) \quad (3.6)$$

where the function c is chosen such that $\partial_z c = b$ on $I \times \mathbb{R}$. Then j is C^1 on $I \times \mathbb{R} \times \mathbb{R}$, $j_p \in C^1(I \times \mathbb{R} \times \mathbb{R})$ and

$$j_{pp}(x, z, p) = \frac{f(\sqrt{x^2 + z^2})}{(1 + p^2)^{3/2}} > 0 \text{ for each } (x, z, p) \in I \times \mathbb{R} \times \mathbb{R}.$$

Then (3.5) may be written

$$0 = \int_I \left\{ j_z(x, u, u')\zeta + j_p(x, u, u')\zeta' \right\} dx$$

for each $\zeta \in C_c^1(I)$. By [5] Proposition 1.9 and Proposition 4.2, $u \in C^2(I)$. \square

Let E be an open set in \mathbb{R}^2 with C^1 boundary M . The inner unit normal vector field on M is denoted n . Assume that $M \setminus \{0\}$ is C^2 . The (mean) curvature of M on $M \setminus \{0\}$ with respect to n is given by

$$H(\cdot, E) := -\operatorname{div}^M n. \quad (3.7)$$

The generalised mean curvature of M at $x \in M \setminus \{0\}$ with respect to n is defined by

$$H_f(x, E) := H(x, E) - (\partial_n \log f)(x) = H(x, E) - \varrho(|x|)\langle n(x), x/|x| \rangle. \quad (3.8)$$

Theorem 3.6. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Let $v > 0$ and E a minimiser of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 and $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 . Then $H_f(\cdot, E)$ is constant on $M \setminus \{0\}$.*

Proof. Let $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with the property that $\operatorname{supp}[X] \cap M \subset M \setminus \{0\}$. By the divergence theorem on manifolds (cf. [1] Theorem 7.34),

$$\begin{aligned} \int_M \langle \nabla f, X \rangle + f \operatorname{div}^M X \, d\mathcal{H}^1 &= \int_M \partial_n f \langle n, X \rangle + \langle \nabla^M f, X \rangle + f \operatorname{div}^M X \, d\mathcal{H}^1 \\ &= \int_M \partial_n f \langle n, X \rangle + \operatorname{div}^M(fX) \, d\mathcal{H}^1 \\ &= \int_M \partial_n f \langle n, X \rangle - H f \langle n, X \rangle \, d\mathcal{H}^1 \\ &= \int_M f u \{ \partial_n \log f - H \} \, d\mathcal{H}^1 \end{aligned}$$

where $u = \langle n, X \rangle$. Combining this with Proposition 3.4 there exists $\lambda \in \mathbb{R}$ such that

$$\int_M u f \{ \partial_n \log f - H + \lambda \} \, d\mathcal{H}^1 = 0$$

for all $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[X] \cap M \subset M \setminus \{0\}$. This leads to the result. \square

Before we turn to a discussion of spherical cap symmetrisation we introduce some notation. Denote by \mathbb{S}_τ^1 the centred circle in \mathbb{R}^2 with radius $\tau > 0$. We sometimes write \mathbb{S}^1 for \mathbb{S}_1^1 . Given $x \in \mathbb{R}^2$, $v \in \mathbb{S}^1$ and $\alpha \in (0, \pi]$ the open cone with vertex x , axis v and opening angle 2α is the set

$$C(x, v, \alpha) := \left\{ y \in \mathbb{R}^2 : \langle y - x, v \rangle > |y - x| \cos \alpha \right\}.$$

Let E be an \mathcal{L}^2 -measurable set in \mathbb{R}^2 and $\tau > 0$. The τ -section E_τ of E is the set $E_\tau := E \cap \mathbb{S}_\tau^1$. Put

$$L(\tau) := \mathcal{H}^1(E_\tau) \text{ for } \tau > 0 \quad (3.9)$$

and $p(E) := \{\tau > 0 : L(\tau) > 0\}$. The function L is \mathcal{L}^1 -measurable by [1] Theorem 2.93. Given $\tau > 0$ and $0 < \alpha \leq \pi$ the spherical cap $C(\tau, \alpha)$ is the set

$$C(\tau, \alpha) := \begin{cases} \mathbb{S}_\tau^1 \cap C(0, e_1, \alpha) & \text{if } 0 < \alpha < \pi; \\ \mathbb{S}_\tau^1 & \text{if } \alpha = \pi; \end{cases}$$

and has \mathcal{H}^1 -measure $s(\tau, \alpha) := 2\alpha\tau$. The spherical cap symmetral E^{sc} of the set E is defined by

$$E^{sc} := \bigcup_{\tau \in p(E)} C(\tau, \alpha) \quad (3.10)$$

where $\alpha \in (0, \pi]$ is determined by $s(\tau, \alpha) = L(\tau)$. Observe that E^{sc} is a \mathcal{L}^2 -measurable set in \mathbb{R}^2 and $V_f(E^{sc}) = V_f(E)$. Note also that if B is a centred open ball then $B^{sc} = B \setminus \{0\}$. We say that E is spherical cap symmetric if $\mathcal{H}^1((E \Delta E^{sc})_\tau) = 0$ for each $\tau > 0$. This definition is broad but suits our purposes.

The result below is stated in [20] Theorem 6.2 and a sketch proof given. A proof along the lines of [2] Theorem 1.1 can be found in [21]. First, let B be a Borel set in $(0, +\infty)$; then the annulus $A(B)$ over B is the set $A(B) := \{x \in \mathbb{R}^2 : |x| \in B\}$.

Theorem 3.7. *Let E be a set of finite perimeter in \mathbb{R}^2 . Then E^{sc} is a set of finite perimeter and*

$$P(E^{sc}, A(B)) \leq P(E, A(B)) \quad (3.11)$$

for any Borel set $B \subset (0, \infty)$ and the same inequality holds with E^{sc} replaced by any set F that is \mathcal{L}^2 -equivalent to E^{sc} .

Corollary 3.8. *Let f be a positive lower semi-continuous radial function on \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 . Then $P_f(E^{sc}) \leq P_f(E)$.*

Proof. Assume that $P_f(E) < +\infty$. We remark that f is Borel measurable as f is lower semi-continuous. Let (f_h) be a sequence of simple Borel measurable radial functions on \mathbb{R}^2 such that $0 \leq f_h \leq f$ and $f_h \uparrow f$ on \mathbb{R}^2 as $h \rightarrow \infty$. By Theorem 3.7,

$$P_{f_h}(E^{sc}) = \int_{\mathbb{R}^2} f_h d|D\chi_{E^{sc}}| \leq \int_{\mathbb{R}^2} f_h d|D\chi_E| = P_{f_h}(E)$$

for each h . Taking the limit $h \rightarrow \infty$ the monotone convergence theorem gives $P_f(E^{sc}) \leq P_f(E)$. \square

Lemma 3.9. *Let E be a \mathcal{L}^2 -measurable set in \mathbb{R}^2 . Then*

- (i) *if $\tau > 0$ and $x \in ((E^{sc})^1)_\tau$ then $y \in (E^{sc})^1_\tau$ for any $y \in \mathbb{S}_\tau^1$ with $|\theta(y)| \leq |\theta(x)|$;*
- (ii) *$(E^{sc})^1$ is spherical cap symmetric;*

(iii) if E is spherical cap symmetric then E^1 is spherical cap symmetric;

(iv) if E is open and spherical cap symmetric then $E \setminus \{0\} = E^{sc}$.

Proof. (i) The case $|\theta(y)| = \pi$ is trivial so we may assume that $|\theta(y)| < \pi$. Assume for a contradiction that for some $\tau > 0$ and $x \in ((E^{sc})^1)_\tau$ there exists $y \notin ((E^{sc})^1)_\tau$ with $|\theta(y)| \leq |\theta(x)|$. There exists $\eta \in (0, 1)$ and a sequence $r_h \downarrow 0$ as $h \rightarrow \infty$ such that

$$\frac{|E^{sc} \cap B(x, r_h)|}{|B(x, r_h)|} \rightarrow 1 \text{ as } h \rightarrow \infty \text{ and } \frac{|B(y, r_h) \setminus E^{sc}|}{|B(y, r_h)|} \geq \eta \text{ for each } h.$$

Choose a rotation $O \in \text{SO}(2)$ such that $OB(x, r_h) = B(y, r_h)$ for each h . Then

$$\liminf_{h \rightarrow \infty} \frac{|O(E^{sc} \cap B(x, r_h)) \cap (B(y, r_h) \setminus E^{sc})|}{|B(y, r_h)|} \geq \eta > 0$$

and so there exists $z \in E^{sc}$ such that $Oz \notin E^{sc}$. This contradicts the definition of E^{sc} . (ii) From (i) the set $[(E^{sc})^1]_\tau$ is an open or closed spherical arc containing $(\tau, 0)$ in \mathbb{S}_τ^1 which is symmetric in the x_1 -axis for each $\tau > 0$. This implies that $(E^{sc})^1$ is spherical cap symmetric. (iii) For any $\tau > 0$,

$$\mathcal{H}^1([E^1 \Delta (E^1)^{sc}]_\tau) = \mathcal{H}^1([(E^{sc})^1 \Delta ((E^{sc})^1)^{sc}]_\tau) = \mathcal{H}^1([(E^{sc})^1 \Delta (E^{sc})^1]_\tau) = 0$$

by (ii). (iv) Let $\tau > 0$. First suppose that $\tau \notin p(E)$. Then $E_\tau = \emptyset$ as E is open. Now suppose that $\tau \in p(E)$. As E is spherical cap symmetric $\mathcal{H}^1([E \Delta E^{sc}]_\tau) = 0$. Now both E and E^{sc} are open in \mathbb{S}_τ^1 . So $[E \Delta E^{sc}]_\tau = \emptyset$. \square

Theorem 3.10. Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Given $v > 0$ there exists a minimiser E of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 , $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$.

Proof. Let E be a bounded minimiser for (1.2). By Corollary 3.8 we may assume that $E = E^{sc}$. By Theorem 3.5, E^1 is open, $M := \partial E^1$ is a C^1 hypersurface in \mathbb{R}^2 and $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 . Finally, $E^1 \setminus \{0\} = (E^1)^{sc}$ by Lemma 3.9. \square

4 σ and its properties

Let E be an open set in \mathbb{R}^2 with C^1 boundary M . Denote by $n : M \rightarrow \mathbb{S}^1$ the inner unit normal vector field. Given $p \in M$ choose a unit vector $t(p) \in \mathbb{S}^1$ in such a way that the pair $\{t(p), n(p)\}$ forms a positively oriented basis for \mathbb{R}^2 . If $p \in M \setminus \{0\}$ let $\sigma(p)$ stand for the angle measured anti-clockwise from the position vector p to the tangent vector $t(p)$; $\sigma(p)$ is uniquely determined up to integer multiples of 2π .

Suppose that M is C^2 in a neighbourhood of $p \in M$. There exists a regular C^2 curve $\gamma_1 : I \rightarrow M$ with $I = (-\delta, \delta)$ for some $\delta > 0$ such that $\gamma_1(0) = p$. We may assume that γ_1 is parametrised by arc-length s . The unit tangent vector is denoted $t_1(s)$ and the unit normal vector $n_1(s)$ is chosen in such a way that the pair $\{t_1(s), n_1(s)\}$ forms a positively oriented basis for \mathbb{R}^2 . We may choose the parametrisation in such a way that $n \circ \gamma_1 = n_1$ on I . The curvature $k_1(s)$ is then defined via the relation

$$t'_1 = k_1 n_1 \text{ on } I \tag{4.1}$$

as in [9] 1.5 for example. The curvature k of M at p is defined by

$$k(p) := k_1(0). \tag{4.2}$$

We note that $k_1 = -\langle n'_1, t_1 \rangle$ on I and in particular,

$$k = H(\cdot, E)$$

at p with $H(\cdot, E)$ as in (3.7) upon making use of (3.1) and (3.2). The generalised mean curvature (3.8) at p may be written in the form

$$H_f(\cdot, E) = k + \varrho(r) \sin \sigma \quad (4.3)$$

in a neighbourhood of p .

Let α_1 stand for the angle measured anti-clockwise from the fixed vector e_1 to the tangent vector t_1 ; α_1 is uniquely determined up to integer multiples of 2π . We may choose α_1 in such a way that $\alpha_1 \in C^1(I)$. Then

$$k_1 = \alpha'_1 \quad (4.4)$$

on I . In case $\gamma_1 \neq 0$ let θ_1 stand for the angle measured anti-clockwise from e_1 to the position vector γ_1 and σ_1 for the angle measured anti-clockwise from the position vector γ_1 to the tangent vector t_1 . By choosing an appropriate branch we may assume that

$$\alpha_1 = \theta_1 + \sigma_1 \quad (4.5)$$

on I and that $\theta_1, \sigma_1 \in C^1(I)$. We may choose σ in such a way that $\sigma \circ \gamma_1 = \sigma_1$ on I . Put $r_1 := |\gamma_1|$ on I ; then $r_1 \in C^1(I)$. It holds that

$$r'_1 = \cos \sigma_1; \quad (4.6)$$

$$r_1 \theta'_1 = \sin \sigma_1; \quad (4.7)$$

on I provided that $r_1 \neq 0$.

Let

$$H := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$$

stand for the open upper half-plane in \mathbb{R}^2 and

$$S : \mathbb{R}^2 \rightarrow \mathbb{R}^2; x = (x_1, x_2) \mapsto (x_1, -x_2)$$

for reflection in the x_1 -axis. Let $O \in \text{SO}(2)$ represent rotation anti-clockwise through $\pi/2$.

Lemma 4.1. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Let $x \in M \setminus \{0\}$. Then*

$$(i) \quad Sx \in M \setminus \{0\};$$

$$(ii) \quad n(Sx) = Sn(x);$$

$$(iii) \quad \cos \sigma(Sx) = -\cos \sigma(x).$$

Proof. (i) The closure \overline{E} of E is spherical cap symmetric. The spherical cap symmetral \overline{E} is invariant under S from the representation (3.10). (ii) is a consequence of this last observation. (iii) Note that $t(Sx) = O^*n(Sx) = O^*Sn(x)$. Then

$$\begin{aligned} \cos \sigma(Sx) &= \langle Sx, t(Sx) \rangle = \langle Sx, O^*Sn(x) \rangle = \langle x, SO^*Sn(x) \rangle \\ &= \langle x, On(x) \rangle = -\langle x, O^*n(x) \rangle = \cos \sigma(x) \end{aligned}$$

as $SO^*S = O$ and $O = -O^*$. □

We introduce the projection $\pi : \mathbb{R}^2 \rightarrow [0, +\infty); x \mapsto |x|$.

Lemma 4.2. *Let E be an open set in \mathbb{R}^2 with boundary M and assume that $E \setminus \{0\} = E^{sc}$.*

- (i) *Suppose $0 \neq x \in \mathbb{R}^2 \setminus \overline{E}$ and $\theta(x) \in (0, \pi]$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (0, \theta(x))$ such that $A(I) \setminus \overline{S}(\alpha) \subset \mathbb{R}^2 \setminus \overline{E}$.*
- (ii) *Suppose $0 \neq x \in \text{int}(E)$ and $\theta(x) \in [0, \pi)$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (\theta(x), \pi)$ such that $A(I) \cap S(\alpha) \subset E$.*
- (iii) *For each $0 < \tau \in \pi(M)$, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 symmetric about the x_1 -axis.*

Proof. (i) We can find $\alpha \in (0, \theta(x))$ such that $\mathbb{S}_\tau^1 \setminus S(\alpha) \subset \mathbb{R}^2 \setminus \overline{E}$ as can be seen from definition (3.10). This latter set is compact so $\text{dist}(\mathbb{S}_\tau^1 \setminus S(\alpha), \overline{E}) > 0$. This means that the ε -neighbourhood of $\mathbb{S}_\tau^1 \setminus S(\alpha)$ is contained in $\mathbb{R}^2 \setminus \overline{E}$ for $\varepsilon > 0$ small. The claim follows. (ii) Again from (3.10) we can find $\alpha \in (\theta(x), \pi)$ such that $\mathbb{S}_\tau^1 \cap S(\alpha) \subset E$ and the assertion follows as before.

(iii) Suppose x_1, x_2 are distinct points in M_τ with $0 \leq \theta(x_1) < \theta(x_2) \leq \pi$. Suppose y lies in the interior of the spherical arc joining x_1 and x_2 . If $y \in \mathbb{R}^2 \setminus \overline{E}$ then $x_2 \in \mathbb{R}^2 \setminus \overline{E}$ by (i) and hence $x_2 \notin M$. If $y \in E$ we obtain the contradiction that $x_1 \in E$ by (ii). Therefore $y \in M$. We infer that the closed spherical arc joining x_1 and x_2 lies in M_τ . The claim follows noting that M_τ is closed. \square

Lemma 4.3. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M . Let $x \in M$. Then*

$$\liminf_{E \ni y \rightarrow x} \left\langle \frac{y-x}{|y-x|}, n(x) \right\rangle \geq 0.$$

Proof. Assume for a contradiction that

$$\liminf_{E \ni y \rightarrow x} \left\langle \frac{y-x}{|y-x|}, n(x) \right\rangle \in [-1, 0).$$

There exists $\eta \in (0, 1)$ and a sequence (y_h) in E such that $y_h \rightarrow x$ as $h \rightarrow \infty$ and

$$\left\langle \frac{y_h - x}{|y_h - x|}, n(x) \right\rangle < -\eta \tag{4.8}$$

for each $h \in \mathbb{N}$. Choose $\alpha \in (0, \pi/2)$ such that $\cos \alpha = \eta$. As M is C^1 there exists $r > 0$ such that

$$B(x, r) \cap C(x, -n(x), \alpha) \cap E = \emptyset.$$

By choosing h sufficiently large we can find $y_h \in B(x, r)$ with the additional property that $y_h \in C(x, -n(x), \alpha)$ by (4.8). We are thus led to a contradiction. \square

Lemma 4.4. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. For each $0 < \tau \in \pi(M)$,*

- (i) *$|\cos \sigma|$ is constant on M_τ ;*
- (ii) *$\cos \sigma = 0$ on $M_\tau \cap \{x_2 = 0\}$;*
- (iii) *$\langle Ox, n(x) \rangle \leq 0$ for $x \in M_\tau \cap H$*
- (iv) *$\cos \sigma \leq 0$ on $M_\tau \cap H$;*

and if $\cos \sigma \not\equiv 0$ on M_τ then

- (v) *$\tau \in p(E)$;*
- (vi) *M_τ consists of two disjoint singletons in \mathbb{S}_τ^1 symmetric about the x_1 -axis;*

(vii) $L(\tau) \in (0, 2\pi\tau)$;

(viii) $M_\tau = \{(\tau \cos(L(\tau)/2\tau), \pm \tau \sin(L(\tau)/2\tau))\}$.

Proof. (i) By Lemma 4.2, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 symmetric about the x_1 -axis. In case $M_\tau \cap \overline{H}$ consists of a singleton the assertion follows from Lemma 4.1. Now suppose that $M_\tau \cap \overline{H}$ consists of a spherical arc in \mathbb{S}_τ^1 with non-empty interior. It can be seen that $\cos \sigma$ vanishes on the interior of this arc as $0 = r'_1 = \cos \sigma_1$ in a local parametrisation by (4.6). By continuity $\cos \sigma = 0$ on M_τ . (ii) follows from Lemma 4.1. (iii) Let $x \in M_\tau \cap H$ so $\theta(x) \in (0, \pi)$. Then $S(\theta(x)) \cap \mathbb{S}_\tau^1 \subset \overline{E}$ as \overline{E} is spherical cap symmetric. Then

$$0 \leq \lim_{S(\theta(x)) \cap \mathbb{S}_\tau^1 \ni y \rightarrow x} \left\langle \frac{y-x}{|y-x|}, n(x) \right\rangle = -\langle Ox, n(x) \rangle$$

by Lemma 4.3. (iv) The adjoint transformation O^* represents rotation clockwise through $\pi/2$. Let $x \in M_\tau \cap H$. By (iii),

$$0 \geq \langle Ox, n(x) \rangle = \langle x, O^*n(x) \rangle = \langle x, t(x) \rangle = \tau \cos \sigma(x)$$

and this leads to the result. (v) As $\cos \sigma \not\equiv 0$ on M_τ we can find $x \in M_\tau \cap H$. We claim that $\mathbb{S}_\tau^1 \cap S(\theta(x)) \subset E$. For suppose that $y \in \mathbb{S}_\tau^1 \cap S(\theta(x))$ but $y \notin E$. We may suppose that $0 \leq \theta(y) < \theta(x) < \pi$. If $y \in \mathbb{R}^2 \setminus \overline{E}$ then $x \in \mathbb{R}^2 \setminus \overline{E}$ by Lemma 4.2. On the other hand, if $y \in M$ then the spherical arc in H joining y to x is contained in M again by Lemma 4.2. This arc also has non-empty interior in \mathbb{S}_τ^1 . Now $\cos \sigma = 0$ on its interior so $\cos(\sigma(x)) = 0$ by (i) contradicting the hypothesis. A similar argument deals with (vi) and this together with (v) in turn entails (vii) and (viii). \square

Lemma 4.5. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Suppose that $0 \in M$. Then*

(i) $(\sin \sigma)(0+) = 0$;

(ii) $(\cos \sigma)(0+) = -1$.

Proof. (i) Let γ_1 be a C^1 parametrisation of M in a neighbourhood of 0 with $\gamma_1(0) = 0$ as above. Then $n(0) = n_1(0) = e_1$ and hence $t(0) = t_1(0) = -e_2$. By Taylor's Theorem $\gamma_1(s) = \gamma_1(0) + t_1(0)s + o(s) = -e_2s + o(s)$ for $s \in I$. This means that $r_1(s) = |\gamma_1(s)| = s + o(s)$ and

$$\cos \theta_1 = \frac{\langle e_1, \gamma_1 \rangle}{r_1} = \frac{\langle e_1, \gamma_1 \rangle}{s} \frac{s}{r_1} \rightarrow 0$$

as $s \rightarrow 0$ which entails that $(\cos \theta_1)(0-) = 0$. Now t_1 is continuous on I so $t_1 = -e_2 + o(1)$ and $\cos \alpha_1 = \langle e_1, t_1 \rangle = o(1)$. We infer that $(\cos \alpha_1)(0-) = 0$. By (4.5), $\cos \alpha_1 = \cos \sigma_1 \cos \theta_1 - \sin \sigma_1 \sin \theta_1$ on I and hence $(\sin \sigma_1)(0-) = 0$. We deduce that $(\sin \sigma)(0+) = 0$. Item (ii) follows from (i) and Lemma 4.4. \square

The set

$$\Omega := \pi \left[(M \setminus \{0\}) \cap \{\cos \sigma \neq 0\} \right] \tag{4.9}$$

plays an important rôle in the proof of Theorem 1.1.

Lemma 4.6. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Then Ω is an open set in $(0, +\infty)$.*

Proof. Suppose $0 < \tau \in \Omega$. Choose $x \in M_\tau \cap \{\cos \sigma \neq 0\}$. Let $\gamma_1 : I \rightarrow M$ be a local C^1 parametrisation of M in a neighbourhood of x such that $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $r_1 \neq 0$ and $\cos \sigma_1 \neq 0$ on I . Then the set $\{r_1(s) : s \in I\} \subset \Omega$

is connected and so an interval in \mathbb{R} (see for example [23] Theorems 6.A and 6.B). By (4.6), $r_1'(0) = \cos \sigma_1(0) = \cos \sigma(p) \neq 0$. This means that the set $\{r_1(s) : s \in I\}$ contains an open interval about τ . \square

Bearing in mind Lemma 4.4 we may define

$$\begin{aligned} \theta_2 : \Omega &\rightarrow (0, \pi); \tau \mapsto (1/2\tau)L(\tau); \\ \gamma : \Omega &\rightarrow M; \tau \mapsto (\tau \cos \theta_2(\tau), \tau \sin \theta_2(\tau)). \end{aligned} \quad (4.10)$$

The function

$$y : \Omega \rightarrow [-1, 1]; \tau \mapsto \sin(\sigma(\gamma(\tau))). \quad (4.11)$$

plays a key role.

Theorem 4.7. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Let $v > 0$ and E a minimiser of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 , $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. Then $y \in C^1(\Omega)$ and there exists $\lambda \in \mathbb{R}$ such that*

$$y' + (1/\tau + \varrho)y + \lambda = 0$$

on Ω .

Proof. Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a C^2 parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. Put $y_1 := \sin \sigma_1$ on I . By shrinking the open interval I if necessary we may assume that $r_1 : I \rightarrow r_1(I)$ is a diffeomorphism and that $r_1(I) \subset \Omega$. Note that $\gamma = \gamma_1 \circ r_1^{-1}$ and $y = y_1 \circ r_1^{-1}$ on $r_1(I)$. It follows that $y \in C^1(\Omega)$. By (4.6),

$$y' = (y_1' \circ r_1^{-1})(r_1^{-1})' = [\cos \sigma_1 \sigma_1'] \circ r_1^{-1} \frac{1}{\cos \sigma_1 \circ r_1^{-1}} = \sigma_1' \circ r_1^{-1}$$

on $r_1(I)$. By (4.4) and (4.7) $\sigma_1' = \alpha_1' - \theta_1' = k_1 - (1/r_1) \sin \sigma_1$. Thus,

$$y' = k - (1/\tau) \sin(\sigma \circ \gamma) = k - (1/\tau)y$$

at τ . By Theorem 3.6 and (4.3) $k + \varrho \sin \sigma$ has constant value $\lambda \in \mathbb{R}$ (say) on $\gamma(\Omega)$. So

$$y' = -\varrho(r)y + \lambda - (1/\tau)y = -(1/r + \varrho(r))y + \lambda$$

on $r_1(I)$. \square

Lemma 4.8. *Assume that f is a positive radial locally Lipschitz density on \mathbb{R}^2 which diverges to infinity. Suppose in addition that f is C^1 on $\mathbb{R}^2 \setminus \{0\}$. Let $v > 0$ and E a minimiser of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 , $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. Then*

- (i) $\theta_2 \in C^1(\Omega)$;
- (ii) $\theta_2' = -\frac{1}{\tau} \frac{y}{\sqrt{1-y^2}}$ on Ω .

Proof. Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a C^2 parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. By shrinking the open interval I if necessary we may assume that $r_1 : I \rightarrow r_1(I)$ is a diffeomorphism and that $r_1(I) \subset \Omega$. It then holds that

$$\theta_2 = \theta_1 \circ r_1^{-1} \text{ and } \sigma \circ \gamma = \sigma_1 \circ r_1^{-1}$$

on $r_1(I)$ by choosing an appropriate branch of θ_1 . It follows that $\theta_2 \in C^1(\Omega)$. By the chain-rule, (4.7) and (4.6),

$$\theta'_2 = (\theta'_1 \circ r_1^{-1})(r_1^{-1})' = [(1/r_1) \tan(\sigma_1)] \circ r_1^{-1} = (1/\tau) \tan(\sigma \circ \gamma)$$

on $r_1(I)$. By Lemma 4.4, $\cos(\sigma \circ \gamma) = -\sqrt{1-y^2}$ on Ω . This entails (ii). \square

Lemma 4.9. *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 . Put $d := \sup\{|x| : x \in M\} > 0$ and choose $b \in M$ such that $d = |b|$. Then $H(b, E) \geq 1/d$.*

Proof. As \overline{E} is spherical cap symmetric we may assume that $b = (d, 0)$. Take a C^2 -parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of b so that $\gamma_1(0) = b$. Assume for a moment that $H(b, E) = 0$. Then $\gamma_1(s) = b + se_2 + R_2(s)$ for $s \in I$ with $|R_2(s)| = o(s^2)$. As a consequence $(r_1^2 - d^2)/s^2 \rightarrow 1$ as $s \rightarrow 0$. But $r_1 \leq d$ on I . This argument shows that $H(b, E) \neq 0$. As M is symmetric in the x_1 -axis we may assume that $\gamma_1(-s) = (S\gamma_1)(s)$ for $s \in I$. We remark that for z in the complex unit disc \mathbb{D} the points $\{1, z, \bar{z}\}$ span a circle with radius at most 1. This is true for $z \in \mathbb{D}$ with $\Re z = 0$; it follows for arbitrary $z \in \mathbb{D}$ by applying a suitable automorphism of \mathbb{D} . An upshot of this remark is that the circle spanning $\{b, \gamma_1(s), \gamma_1(-s)\}$ has radius at most d . A compactness argument as well as [24] Theorem 1.1 entails that $H(b, E) \geq 1/d$. \square

Given f as in (1.3) put $\varrho := h'$ on $(0, +\infty)$.

Theorem 4.10. *Let f be as in (1.3). Let $v > 0$ and E a minimiser of (1.2) such that E is open with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 . Then E is convex.*

Proof. The proof runs along the same lines as [20] Theorem 6.5. Let $b \in M$ as above. Then $d > 0$ and $\sin \sigma(b) = 1$. The expression $k + \varrho \sin \sigma$ is constant on $M \setminus \{0\}$ by (4.3) and Theorem 3.6. By monotonicity of ϱ , $\varrho \sin \sigma \leq \varrho \leq \varrho(b)$ on $M \setminus \{0\}$ so $k \geq k(b) \geq 1/d > 0$ on $M \setminus \{0\}$ by Lemma 4.9. The set E is then convex by a small modification of [24] Theorem 1.8 and Proposition 1.4. It is sufficient that the function f (here α_1) in the proof of the former theorem is non-decreasing. \square

5 A reverse Hermite-Hadamard inequality

Let $0 \leq a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be monotone increasing. Let h be a primitive of ϱ on (a, b) so that $h \in C([a, b]) \cap C^1((a, b))$ and introduce the functions

$$\mathbf{f} : [a, b] \rightarrow \mathbb{R}; x \mapsto e^{h(x)}; \quad (5.1)$$

$$\varrho : [a, b] \rightarrow \mathbb{R}; x \mapsto x\mathbf{f}(x). \quad (5.2)$$

Then

$$g' = (1/x + \varrho)g = \mathbf{f} + g\varrho \quad (5.3)$$

on (a, b) . Define

$$m = m(\varrho, a, b) := \frac{g(b) - g(a)}{\int_a^b g \, dt}. \quad (5.4)$$

If ϱ takes the constant value $\mathbb{R} \ni \lambda \geq 0$ on $[a, b]$ we use the notation $m(\lambda, a, b)$ and we write $m_0 = m(0, a, b)$. A computation gives

$$m_0 = m(0, a, b) = A(a, b)^{-1} \quad (5.5)$$

where $A(a, b) := (a + b)/2$ stands for the arithmetic mean of a and b .

Lemma 5.1. Let $0 \leq a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be monotone increasing. Then $m_0 \leq m$.

Proof. Note that g is convex on $[a, b]$ as g' is non-decreasing as can be seen from (5.3). By the Hermite-Hadamard inequality (cf. [16], [14]),

$$\int_a^b g \, dt \leq (b-a) \frac{g(a) + g(b)}{2}.$$

The algebraic inequality $(b-a)(g(a) + g(b)) \leq (a+b)(g(b) - g(a))$ entails that

$$\int_a^b g \, dt \leq \frac{a+b}{2}(g(b) - g(a))$$

and the result follows on rearrangement. \square

Lemma 5.2. Let $0 \leq a < b < +\infty$ and $\lambda > 0$. Then $m(\lambda, a, b) < \lambda + A(a, b)^{-1}$.

Proof. First suppose that $\lambda = 1$ and take $h : [a, b] \rightarrow \mathbb{R}; t \mapsto t$. In this case,

$$\int_a^b g \, dt = \int_a^b t e^t \, dt = (b-1)e^b - (a-1)e^a$$

and

$$m(1, a, b) = \frac{be^b - ae^a}{(b-1)e^b - (a-1)e^a}.$$

The inequality in the statement is equivalent to

$$(a+b)(be^b - ae^a) < ((b-1)e^b - (a-1)e^a)(2+a+b)$$

which in turn is equivalent to the statement $\tanh[(b-a)/2] < (b-a)/2$ which holds for any $b > a$.

For $\lambda > 0$ take $h : [a, b] \rightarrow \mathbb{R}; t \mapsto \lambda t$. Substitution gives

$$\int_a^b g \, dt = (1/\lambda)^2 [(\lambda b - 1)e^{\lambda b} - (\lambda a - 1)e^{\lambda a}] \text{ and } g(b) - g(a) = (1/\lambda)[\lambda b e^{\lambda b} - \lambda a e^{\lambda a}]$$

so from above

$$m(\lambda, a, b) = \lambda m(1, \lambda a, \lambda b) < \lambda \left\{ 1 + A(\lambda a, \lambda b)^{-1} \right\} = \lambda + A(a, b)^{-1}.$$

\square

Theorem 5.3. Let $0 \leq a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be monotone increasing. Then

- (i) $m(\varrho, a, b) \leq \varrho(b) + A(a, b)^{-1}$;
- (ii) equality holds if and only if $\varrho \equiv 0$ on $[a, b]$.

Proof. (i) Let h be a primitive for ϱ on (a, b) . Define $h_1 : [a, b] \rightarrow \mathbb{R}; t \mapsto h(b) - \varrho(b)(b-t)$. Then $h_1(b) = h(b)$, $h'_1 = \varrho(b) \geq \varrho = h'$ on (a, b) and hence $h \geq h_1$ on (a, b) . We derive that

$$\int_a^b g \, dt = \int_a^b t e^{h(t)} \, dt \geq \int_a^b t e^{h_1(t)} \, dt = \int_a^b g_1 \, dt$$

and

$$g(b) - g(a) = be^{h(b)} - ae^{h(a)} = be^{h_1(b)} - ae^{h_1(a)} \leq be^{h_1(b)} - ae^{h_1(a)} = g_1(b) - g_1(a)$$

with obvious notation. This entails that $m(\varrho, a, b) \leq m(\varrho(b), a, b)$ and the result follows with the help of Lemma 5.2.

(ii) Suppose that $\varrho \not\equiv 0$ on $[a, b]$. If ϱ is constant on $[a, b]$ the assertion follows from Lemma 5.2. Assume then that ϱ is not constant on $[a, b]$. Then $h \not\equiv h_1$ on $[a, b]$ in the above notation and $\int_a^b t e^{h(t)} dt > \int_a^b t e^{h_1(t)} dt$ which entails strict inequality in (i). \square

With the above notation define

$$\hat{m} = \hat{m}(\varrho, a, b) := \frac{g(a) + g(b)}{\int_a^b g dt}. \quad (5.6)$$

A computation gives

$$\hat{m}_0 := \hat{m}(0, a, b) = \frac{2}{b-a}. \quad (5.7)$$

Lemma 5.4. *Let $0 \leq a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be monotone increasing. Then $\hat{m} \geq \hat{m}_0$.*

Proof. This follows by the Hermite-Hadamard inequality (cf. [16], [14]),

$$\int_a^b g dt \leq (b-a) \frac{g(a) + g(b)}{2}.$$

\square

We prove a reverse Hermite-Hadamard inequality.

Theorem 5.5. *Let $-\infty < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ non-decreasing. Then*

- (i) $(b-a)\hat{m}(\varrho, a, b) \leq 2 + a\varrho(a) + b\varrho(b);$
- (ii) *equality holds if and only if $\varrho \equiv 0$ on $[a, b]$.*

Proof. (i) We assume in the first instance that $\varrho \in C^1((a, b))$. We prove the above result in the form

$$\int_a^b g dt \geq (b-a) \frac{g(a) + g(b)}{2 + a\varrho(a) + b\varrho(b)}. \quad (5.8)$$

Put

$$w := \frac{(t-a)(g(a) + g)}{2 + a\varrho(a) + t\varrho}$$

for $t \in [a, b]$ so that

$$\int_a^b w' dt = (b-a) \frac{g(a) + g(b)}{2 + a\varrho(a) + b\varrho(b)}.$$

Then using (5.3),

$$\begin{aligned} w' &= \frac{(g(a) + g + (t-a)g')(2 + a\varrho(a) + t\varrho) - (t-a)(g(a) + g)(\varrho + t\varrho')}{(2 + a\varrho(a) + t\varrho)^2} \\ &= \frac{(g(a) - ag' + (2 + t\varrho)g)(2 + a\varrho(a) + t\varrho) - (t-a)(g(a) + g)(\varrho + t\varrho')}{(2 + a\varrho(a) + t\varrho)^2} \\ &= \frac{(2 + t\varrho)(2 + a\varrho(a) + t\varrho)}{(2 + a\varrho(a) + t\varrho)^2} g + \frac{(g(a) - ag')(2 + a\varrho(a) + t\varrho) - (t-a)(g(a) + g)(\varrho + t\varrho')}{(2 + a\varrho(a) + t\varrho)^2} \\ &\leq g - \frac{2g(a)}{(2 + a\varrho(a) + b\varrho(b))^2} (t-a)\varrho \\ &\leq g \end{aligned} \quad (5.9)$$

on (a, b) as

$$g(a) - ag' = a(\mathbf{f}(a) - (1/t + \varrho)g) = a(\mathbf{f}(a) - \mathbf{f} - \varrho g) \leq 0.$$

An integration over $[a, b]$ gives the result.

Let us assume now that $\varrho \in C([a, b])$. Extend ϱ to \mathbb{R} via

$$\tilde{\varrho}(t) := \begin{cases} \varrho(a) & \text{for } t \in (-\infty, a]; \\ \varrho(t) & \text{for } t \in (a, b]; \\ \varrho(b) & \text{for } t \in (b, +\infty); \end{cases}$$

for $t \in \mathbb{R}$. Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers (see e.g. [1] 2.1) and set $\tilde{\varrho}_\varepsilon := \tilde{\varrho} \star \psi_\varepsilon$ on \mathbb{R} for each $\varepsilon > 0$. Then $\tilde{\varrho}_\varepsilon \in C^\infty(\mathbb{R})$ and is non-decreasing on \mathbb{R} for each $\varepsilon > 0$. Put $\varrho_\varepsilon := \tilde{\varrho}_\varepsilon|_{[a, b]}$ for each $\varepsilon > 0$. Then $(\varrho_\varepsilon)_{\varepsilon>0}$ converges uniformly to ϱ on $[a, b]$ as $\varepsilon \downarrow 0$. By the above result,

$$(b - a)\hat{m}(\varrho_\varepsilon, a, b) \leq 2 + a\varrho_\varepsilon(a) + b\varrho_\varepsilon(b)$$

for each $\varepsilon > 0$. The inequality follows on taking the limit $\varepsilon \downarrow 0$.

(ii) We now consider the equality case. We claim that

$$(b - a) \frac{g(a) + g(b)}{2 + a\varrho(a) + b\varrho(b)} \leq \int_a^b g \, dt - \frac{2g(a)}{(2 + a\varrho(a) + b\varrho(b))^2} \int_a^b (t - a)\varrho \, dt; \quad (5.10)$$

this entails the equality condition in (ii). First suppose that $\varrho \in C^1((a, b))$. In this case the inequality in (5.9) implies (5.10) upon integration. Now suppose that $\varrho \in C([a, b])$. Then (5.10) holds with ϱ_ε in place of ϱ for each $\varepsilon > 0$. The inequality then follows because ϱ is the uniform limit of $(\varrho_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$. \square

6 Comparison theorems for first-order differential equations

Let \mathcal{L} stand for the collection of Lebesgue measurable sets in $[0, +\infty)$. Define a measure μ on $([0, +\infty), \mathcal{L})$ by $\mu(dx) := (1/x)dx$. Let $0 \leq a < b < +\infty$. Suppose that $u : [a, b] \rightarrow \mathbb{R}$ is an \mathcal{L}^1 -measurable function with the property that

$$\mu(\{u > t\}) < +\infty \text{ for each } t > 0. \quad (6.1)$$

The distribution function $\mu_u : (0, +\infty) \rightarrow [0, +\infty)$ of u with respect to μ is given by

$$\mu_u(t) := \mu(\{u > t\}) \text{ for } t > 0.$$

Note that μ_u is right-continuous and non-increasing on $(0, \infty)$ and $\mu_u(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let u be a C^1 function on $[a, b]$. Put $Z := \{u' = 0\}$. By [1] Lemma 2.96, $Z \cap \{u = t\} = \emptyset$ for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is \mathcal{L}^1 -negligible. We make use of the coarea formula ([1] Theorem 2.93 and (2.74)),

$$\int_{[a, b]} \phi |u'| \, dx = \int_{-\infty}^{\infty} \int_{\{u=t\}} \phi \, d\mathcal{H}^0 \, dt \quad (6.2)$$

for any \mathcal{L}^1 -measurable function $\phi : [a, b] \rightarrow [0, \infty]$.

Lemma 6.1. *Let $0 \leq a < b < +\infty$ and u a C^1 function on $[a, b]$. Then*

- (i) $\mu_u \in \text{BV}_{\text{loc}}((0, +\infty))$;
- (ii) $D\mu_u = -u_\# \mu$;
- (iii) $D\mu_u^a = D\mu_u \llcorner ((0, +\infty) \setminus N)$;

(iv) $D\mu_u^s = D\mu_u \llcorner N$;

(v) $A := \left\{ t \in (0, +\infty) : \mathcal{L}^1(Z \cap \{u = t\}) > 0 \right\}$ is the set of atoms of $D\mu_u$ and $D\mu_u^j = D\mu_u \llcorner A$;

(vi) μ_u is differentiable \mathcal{L}^1 -a.e. on $(0, +\infty)$ with derivative given by

$$\mu'_u(t) = - \int_{\{u=t\} \setminus Z} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}$$

for \mathcal{L}^1 -a.e. $t \in (0, +\infty)$;

(vii) $\text{Ran}(u) \cap [0, +\infty) = \text{supp}(D\mu_u)$.

The notation above $D\mu_u^a$, $D\mu_u^s$, $D\mu_u^j$ stands for the absolutely continuous resp. singular resp. jump part of the measure $D\mu_u$ (see [1] 3.2 for example).

Proof. For any $\varphi \in C_c^\infty((0, +\infty))$ with $\text{supp}[\varphi] \subset (\tau, +\infty)$ for some $\tau > 0$,

$$\int_0^\infty \mu_u \varphi' dt = \int_{[a,b]} \varphi \circ u d\mu = \int_{[a,b]} \chi_{\{u>\tau\}} \varphi \circ u d\mu \quad (6.3)$$

by Fubini's theorem; so $\mu_u \in \text{BV}_{\text{loc}}((0, +\infty))$ and $D\mu_u$ is the push-forward of μ under u , $D\mu_u = -u_\# \mu$ (cf. [1] 1.70). By (6.2),

$$D\mu_u \llcorner ((0, +\infty) \setminus N)(A) = -\mu(\{u \in A\} \setminus Z) = - \int_A \int_{\{u=t\} \setminus Z} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau} dt$$

for any \mathcal{L}^1 -measurable set A in $(0, +\infty)$. In light of the above, we may identify $D\mu_u^a = D\mu_u \llcorner ((0, +\infty) \setminus N)$ and $D\mu_u^s = D\mu_u \llcorner N$. The set of atoms of $D\mu_u$ is defined by $A := \{t \in (0, +\infty) : D\mu_u(\{t\}) \neq 0\}$. For $t > 0$,

$$D\mu_u(\{t\}) = D\mu_u^s(\{t\}) = (D\mu_u \llcorner N)(\{t\}) = -u_\# \mu(N \cap \{t\}) = -\mu(Z \cap \{u = t\})$$

and this entails (v). The monotone function μ_u is a good representative within its equivalence class and is differentiable \mathcal{L}^1 -a.e. on $(0, +\infty)$ with derivative given by the density of $D\mu_u$ with respect to \mathcal{L}^1 by [1] Theorem 3.28. Item (vi) follows from (6.2) and (iii). Item (vii) follows from (ii). \square

Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let $\eta \in \{\pm 1\}^2$. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \varrho)u + \lambda = 0 \text{ on } (a, b) \text{ with } u(a) = \eta_1 \text{ and } u(b) = \eta_2 \quad (6.4)$$

where $u \in C([a, b]) \cap C^1((a, b))$ and $\lambda \in \mathbb{R}$. In case $\varrho \equiv 0$ on $[a, b]$ we use the notation u_0 .

Lemma 6.2. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let $\eta \in \{\pm 1\}^2$. Then*

(i) *there exists a solution (u, λ) of (6.4) with $u \in C([a, b]) \cap C^1((a, b))$ and $\lambda = \lambda_\eta \in \mathbb{R}$;*

(ii) *the pair (u, λ) in (i) is unique;*

(iii) *λ_η is given by*

$$-\lambda_{(1,1)} = \lambda_{(-1,-1)} = m; \lambda_{(1,-1)} = -\lambda_{(-1,1)} = \hat{m};$$

(iv) *if $\eta = (1, 1)$ or $\eta = (-1, -1)$ then u is uniformly bounded away from zero on $[a, b]$.*

Proof. (i) For $\eta = (1, 1)$ define $u : [a, b] \rightarrow \mathbb{R}$ by

$$u(t) := \frac{m \int_a^t g ds + g(a)}{g(t)} \text{ for } t \in [a, b] \quad (6.5)$$

with m as in (5.4). Then $u \in C([a, b]) \cap C^1((a, b))$ and satisfies (6.4) with $\lambda = -m$. For $\eta = (1, -1)$ set $u = (-\hat{m} \int_a^t g ds + g(a))/g$ with $\lambda = \hat{m}$. The cases $\eta = (-1, -1)$ and $\eta = (-1, 1)$ can be dealt with using linearity. (ii) We consider the case $\eta = (1, 1)$. Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (6.4). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \varrho)u + \lambda = 0 \text{ on } (a, b) \text{ with } u(a) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda \int_a^t g ds + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. The other cases are similar. (iii) follows as in (i). (iv) If $\eta = (1, 1)$ then $u > 0$ on $[a, b]$ from (6.5) as $m > 0$. \square

The boundary condition $\eta_1 \eta_2 = -1$.

Lemma 6.3. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let (u, λ) solve (6.4) with $\eta = (1, -1)$. Then*

- (i) *there exists a unique $c \in (a, b)$ with $u(c) = 0$;*
- (ii) *$u' < 0$ and u is strictly decreasing on $[a, c]$;*
- (iii) *$D\mu_u^s = 0$.*

Proof. (i) We first observe that $u' \leq -\hat{m} < 0$ on $\{u \geq 0\}$ in view of (6.4). Suppose $u(c_1) = u(c_2)$ for some $c_1, c_2 \in (a, b)$ with $c_1 < c_2$. We may assume that $u \geq 0$ on $[c_1, c_2]$. By Rolle's Theorem there exists $\xi \in (c_1, c_2)$ such that $u'(\xi) = 0$. This contradicts the above observation. Item (ii) is plain. The set $N \cap \{u > 0\} = \emptyset$ by (ii). The assertion in (iii) follows from Lemma 6.1. \square

Lemma 6.4. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let (u, λ) solve (6.4) with $\eta = (1, -1)$. Assume that $u'(a) < 0$ and $u'(b) < 0$. Put $v := -u$. Then*

- (i) $\int_{\{v=1\}} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} \geq \int_{\{u=1\}} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau};$
- (ii) *equality holds if and only if $\varrho \equiv 0$ on $[a, b]$.*

Proof. First, $\{u = 1\} = \{a\}$ by Lemma 6.3. Further $0 < -au'(a) = 1 + a[\hat{m} + \varrho(a)]$ from (6.4). On the other hand $\{v = 1\} \supset \{b\}$ and $0 < bv'(b) = -1 + b[\hat{m} - \varrho(b)]$. Thus

$$\int_{\{v=1\}} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} - \int_{\{u=1\}} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau} \geq \frac{1}{-1 + b[\hat{m} - \varrho(b)]} - \frac{1}{1 + a[\hat{m} + \varrho(a)]}.$$

By Theorem 5.5, $0 \leq 2 + (a - b)\hat{m} + a\varrho(a) + b\varrho(b)$. A rearrangement leads to the inequality. The equality assertion follows from Theorem 5.5. \square

Theorem 6.5. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Suppose that (u, λ) solves (6.4) with $\eta = (1, -1)$ and set $v := -u$. Assume that $u > -1$ on $[a, b]$. Then*

- (i) $-\mu'_v \geq -\mu'_u$ for \mathcal{L}^1 -a.e. $t \in (0, 1)$;
- (ii) if $\varrho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (0, 1)$ such that $-\mu'_v > -\mu'_u$ for \mathcal{L}^1 -a.e. $t \in (t_0, 1)$;
- (iii) for $t \in [-1, 1]$,

$$\mu_{u_0}(t) = \log \left\{ \frac{-(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}}{2a} \right\}$$

and $\mu_{v_0} = \mu_{u_0}$ on $[-1, 1]$;

in obvious notation.

Proof. (i) For each $t \in (0, 1)$ put $c := \max\{u \geq t\}$. Then $c \in (a, b)$, $\{u > t\} = [a, c)$ and $u'(c) < 0$ by Lemma 6.3. As $v \in C^1((a, b))$ the set $\{v = t\} \cap Z_v = \emptyset$ for \mathcal{L}^1 -a.e. $t \in (0, 1)$. Suppose that $t \in (0, 1)$ has this property. Put $d := \max\{v \leq t\} = \max\{u \geq -t\}$. As u is continuous on $[a, b]$ it holds that $a < c < d < b$. Moreover, $u'(d) < 0$ as $v(d) = t$ and $d \notin Z_v$. Put $\tilde{u} := u/t$ and $\tilde{v} := v/t$ on $[c, d]$. Then

$$\begin{aligned} \tilde{u}' + (1/\tau + \varrho)\tilde{u} + \hat{m}/t &= 0 \text{ on } (c, d) \text{ and } \tilde{u}(c) = -\tilde{u}(d) = 1; \\ \tilde{v}' + (1/\tau + \varrho)\tilde{v} - \hat{m}/t &= 0 \text{ on } (c, d) \text{ and } -\tilde{v}(c) = \tilde{v}(d) = 1. \end{aligned}$$

By Lemma 6.4,

$$\begin{aligned} \int_{\{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} &\geq \int_{[c,d] \cap \{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} = (1/t) \int_{[c,d] \cap \{\tilde{v}=1\} \setminus Z_v} \frac{1}{|\tilde{v}'|} \frac{d\mathcal{H}^0}{\tau} \\ &\geq (1/t) \int_{[c,d] \cap \{\tilde{u}=1\} \setminus Z_u} \frac{1}{|\tilde{u}'|} \frac{d\mathcal{H}^0}{\tau} = \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}. \end{aligned}$$

By Lemma 6.1,

$$-\mu'_u(t) = \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}$$

for \mathcal{L}^1 -a.e. $t \in (0, 1)$ and a similar formula holds for v . The assertion in (i) follows.

(ii) Assume that $\varrho \not\equiv 0$ on $[a, b]$. Put $\alpha := \inf\{\varrho > 0\} \in [a, b]$. Note that $\max\{v \leq t\} \rightarrow b$ as $t \uparrow 1$ as $v < 1$ on $[a, b]$ by assumption. Choose $t_0 \in (0, 1)$ such that $\max\{v \leq t_0\} > \alpha$. Then for $t > t_0$,

$$a < \max\{u \geq t\} < \max\{u \geq -t_0\} = \max\{v \leq t_0\} < \max\{v \leq t\} < d;$$

that is, the interval $[c, d]$ with c, d as described above intersects $(\alpha, b]$. So for \mathcal{L}^1 -a.e. $t \in (t_0, 1)$,

$$\int_{\{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} > \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}.$$

by the equality condition in Lemma 6.4. The conclusion follows from the representation of μ_u resp. μ_v in Lemma 6.1.

(iii) A direct computation gives

$$u_0(\tau) = \frac{1}{b-a} \left\{ -\tau + \frac{ab}{\tau} \right\}$$

for $\tau \in [a, b]$; u_0 is strictly decreasing on its domain. This leads to the formula in (iii). A similar computation gives

$$\mu_{v_0}(t) = \log \left\{ \frac{2b}{(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}} \right\}$$

for $t \in [-1, 1]$. Rationalising the denominator results in the stated equality. \square

Corollary 6.6. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Suppose that (u, λ) solves (6.4) with $\eta = (1, -1)$ and set $v := -u$. Assume that $u > -1$ on $[a, b]$. Then*

(i) $\mu_u(t) \leq \mu_v(t)$ for each $t \in (0, 1)$;

(ii) if $\varrho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (0, 1)$ such that $\mu_u(t) < \mu_v(t)$ for each $t \in (t_0, 1)$.

Proof. (i) By [1] Theorem 3.28 and Lemma 6.3,

$$\mu_u(t) = \mu_u(t) - \mu_u(1) = -D\mu_u((t, 1]) = -D\mu_u^a((t, 1]) - D\mu_u^s((t, 1]) = - \int_{(t, 1]} \mu'_u ds$$

for each $t \in (0, 1)$ as $\mu_u(1) = 0$. On the other hand,

$$\mu_v(t) = \mu_v(1) + (\mu_v(t) - \mu_v(1)) = \mu_v(1) - D\mu_v((t, 1]) = \mu_v(1) - \int_{(t, 1]} \mu'_v ds - D\mu_v^s((t, 1])$$

for each $t \in (0, 1)$. The claim follows from Theorem 6.5 noting that $D\mu_v^s((t, 1]) \leq 0$ as can be seen from Lemma 6.1. Item (ii) follows from Theorem 6.5 (ii). \square

Corollary 6.7. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Suppose that (u, λ) solves (6.4) with $\eta = (1, -1)$. Assume that $u > -1$ on $[a, b]$. Let $\varphi \in C^1((-1, 1))$ be an odd non-decreasing bounded function and assume that $\varphi \in L^1((-1, 1))$. Then*

$$(i) \int_{\{u>0\}} \varphi(u) d\mu < +\infty;$$

$$(ii) \int_a^b \varphi(u) d\mu \leq 0;$$

(iii) equality holds in (ii) if and only if $\varrho \equiv 0$ on $[a, b]$.

In particular,

$$(iv) \int_a^b \frac{u}{\sqrt{1-u^2}} d\mu \leq 0 \text{ with equality if and only if } \varrho \equiv 0 \text{ on } [a, b].$$

Proof. (i) Put $I := \{u > 0\}$. The function $u : I \rightarrow (0, 1)$ is C^1 and $u' \leq -\hat{m}$ by Lemma 6.3. It has C^1 inverse $v : (0, 1) \rightarrow I$, $v' = 1/(u' \circ v)$ and $|v'| \leq 1/\hat{m}$. By a change of variables,

$$\int_{\{u>0\}} \varphi(u) d\mu = \int_0^1 \varphi(v'/v) dt$$

from which the claim is apparent. (ii) The integral is well-defined because $\varphi(u)^+ = \varphi(u)\chi_{\{u>0\}} \in L^1((a, b), \mu)$ by (i). By Lemma 6.3 the set $\{u = 0\}$ consists of a singleton and has μ -measure zero. So

$$\int_a^b \varphi(u) d\mu = \int_{\{u>0\}} \varphi(u) d\mu + \int_{\{u<0\}} \varphi(u) d\mu = \int_{\{u>0\}} \varphi(u) d\mu - \int_{\{v>0\}} \varphi(v) d\mu$$

where $v := -u$ as φ is an odd function. We remark that in a similar way to (6.3),

$$\int_0^1 \varphi' \mu_u dt = \int_{\{u>0\}} \left\{ \varphi(u) - \varphi(0) \right\} d\mu = \int_{\{u>0\}} \varphi(u) d\mu$$

using oddness of φ and an analogous formula holds with v in place of u . Thus we may write

$$\int_a^b \varphi(u) d\mu = \int_0^1 \varphi' \mu_u dt - \int_0^1 \varphi' \mu_v dt = \int_0^1 \varphi' \left\{ \mu_u - \mu_v \right\} dt \leq 0$$

by Corollary 6.6 as $\varphi' \geq 0$ on $(0, 1)$. (iii) Suppose that $\varrho \not\equiv 0$ on $[a, b]$. Choose $t_0 \in (0, 1)$ as in Corollary 6.6. The claim is then apparent upon splitting the last integral into a sum of integrals over $[0, t_0]$ resp. $[t_0, 1]$ and using Corollary 6.6. If $\varrho \equiv 0$ on $[a, b]$ the equality follows from Theorem 6.5. (iv) follows from (ii) and (iii) with the particular choice $\varphi : (-1, 1) \rightarrow \mathbb{R}; t \mapsto t/\sqrt{1-t^2}$. \square

The boundary condition $\eta_1 \eta_2 = 1$. Let $0 < a < b < +\infty$ and $\varrho \in C([a, b])$ be non-decreasing. We study solutions of the auxilliary Riccati equation

$$w' + \lambda w^2 = (1/x + \varrho)w \text{ on } (a, b) \text{ with } w(a) = w(b) = 1; \quad (6.6)$$

with $w \in C([a, b]) \cap C^1((a, b))$ and $\lambda \in \mathbb{R}$. If $\varrho \equiv 0$ on $[a, b]$ then we write w_0 instead of w . Suppose (u, λ) solves (6.4) with $\eta = (1, 1)$. Then $u > 0$ on $[a, b]$ by Lemma 6.2 and we may set $w := 1/u$. Then $(w, -\lambda)$ satisfies (6.6).

Lemma 6.8. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Then*

- (i) *there exists a solution (w, λ) of (6.6) with $w \in C([a, b]) \cap C^1([a, b])$ and $\lambda \in \mathbb{R}$;*
- (ii) *the pair (w, λ) in (i) is unique;*
- (iii) *$\lambda = m$.*

Proof. (i) Define $w : [a, b] \rightarrow \mathbb{R}$ by

$$w(t) := \frac{g(t)}{m \int_a^t g \, ds + g(a)} \text{ for } t \in [a, b].$$

Then $w \in C([a, b]) \cap C^1([a, b])$ and satisfies (6.6). (ii) Define $u : [a, b] \rightarrow (0, +\infty)$ by

$$u(t) := \int_a^t \exp \left[\int_a^s \frac{1}{\tau} + \varrho(\tau) \, d\tau \right] ds.$$

Suppose there exists $m_1 \in \mathbb{R}$ and $w_1 \in C([a, b]) \cap C^1([a, b])$ such that

$$w_1' + m_1 w_1^2 = (1/x + \varrho) w_1 \text{ on } (a, b) \text{ with } w_1(a) = w_1(b) = 1.$$

Then $m w = u'/u = m_1 w_1$ on (a, b) and hence $m_1 = m$ and $w_1 = w$ on $[a, b]$. (iv) □

We introduce the mapping

$$\omega : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}; (t, x) \mapsto -(2/t) \coth(x/2).$$

For $\xi > 0$,

$$|\omega(t, x) - \omega(t, y)| \leq (\operatorname{sech}^2 \xi)(1/t)|x - y| \tag{6.7}$$

for $(t, x), (t, y) \in (0, \infty) \times (\xi, \infty)$ and ω is locally Lipschitzian in x on $(0, \infty) \times (0, \infty)$ in the sense of [15] I.3. Let $0 < a < b < +\infty$ and set $\lambda := A/G > 1$. Here, $A = A(a, b)$ stands for the arithmetic mean of a, b as introduced in the previous Section while $G = G(a, b) := \sqrt{|ab|}$ stands for their geometric mean. We refer to the initial value problem

$$z' = \omega(t, z) \text{ on } (0, \lambda) \text{ and } z(1) = \mu((a, b)). \tag{6.8}$$

Define

$$z_0 : (0, \lambda) \rightarrow \mathbb{R}; t \mapsto 2 \log \left\{ \frac{\lambda + \sqrt{\lambda^2 - t^2}}{t} \right\}.$$

Lemma 6.9. *Let $0 < a < b < +\infty$. Then*

- (i) $w_0(\tau) = \frac{2A\tau}{G^2 + \tau^2}$ for $\tau \in [a, b]$;
- (ii) $\|w_0\|_\infty = \lambda$;
- (iii) $\mu_{w_0} = z_0$ on $[1, \lambda]$;
- (iv) z_0 satisfies (6.8) and this solution is unique;
- (v) $\int_{\{w_0=1\}} \frac{1}{|w_0|} \frac{d\mathcal{H}^0}{\tau} = 2 \coth(\mu((a, b))/2)$;
- (vi) $\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{dx}{x} = \pi$.

Proof. (i) follows as in the proof of Lemma 6.8 with $g(t) = t$ while (ii) follows by calculus. (iii) follows by solving the quadratic equation $t\tau^2 - 2A\tau + G^2t = 0$ for τ with $t \in (0, \lambda)$. Uniqueness in (iv) follows from [15] Theorem 3.1 as ω is locally Lipschitzian with respect to x in $(0, \infty) \times (0, \infty)$. For (v) note that $|aw'_0(a)| = 1 - a/A$ and $|bw'_0(b)| = b/A - 1$ and

$$2 \coth(\mu((a, b))/2) = 2(a + b)/(b - a).$$

(vi) We may write

$$\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} = \int_a^b \frac{ab + \tau^2}{\sqrt{(a + b)^2 \tau^2 - (ab + \tau^2)^2}} \frac{d\tau}{\tau} = \int_a^b \frac{ab + \tau^2}{\sqrt{(\tau^2 - a^2)(b^2 - \tau^2)}} \frac{d\tau}{\tau}.$$

The substitution $s = \tau^2$ followed by the Euler substitution (cf. [13] 2.251) $\sqrt{(s - a^2)(b^2 - s)} = t(s - a^2)$ gives

$$\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} = \int_0^\infty \frac{1}{1 + t^2} + \frac{ab}{b^2 + a^2 t^2} dt = \pi.$$

□

Lemma 6.10. *Let $0 < a < b < +\infty$. Then*

(i) *for $y \in (a, b)$ the function $x \mapsto \frac{by - ax}{(y - a)(b - x)}$ is strictly increasing on $[y, b]$;*

(ii) *the function $y \mapsto \frac{(b - a)y}{(y - a)(b - y)}$ is strictly increasing on $[G, b]$;*

(iii) *for $x \leq b$ the function $y \mapsto \frac{by - ax}{y - a}$ is strictly decreasing on \mathbb{R} .*

Proof. The proof is an exercise in calculus. □

Lemma 6.11. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let (w, λ) solve (6.6). Assume*

(i) $w'(a) > 0$ and $w'(b) < 0$;

(ii) $w > 1$ on (a, b) .

Then

$$\int_{\{w=1\}} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{\tau} \geq 2 \coth(\mu((a, b))/2)$$

with equality if and only if $\varrho \equiv 0$ on $[a, b]$.

Proof. At the end-points $x = a, b$ (6.6) entails that $w' + m - \varrho = 1/x = w'_0 + m_0$ so that

$$w' - w'_0 = m_0 - m + \varrho \text{ at } x = a, b. \quad (6.9)$$

We consider the four cases

(a) $w'(a) \geq w'_0(a)$ and $w'(b) \geq w'_0(b)$;

(b) $w'(a) \geq w'_0(a)$ and $w'(b) \leq w'_0(b)$;

(c) $w'(a) \leq w'_0(a)$ and $w'(b) \geq w'_0(b)$;

(d) $w'(a) \leq w'_0(a)$ and $w'(b) \leq w'_0(b)$;

in turn.

(a) Condition (a) together with (6.9) means that $m_0 - m + \varrho(a) \geq 0$; that is, $m - \varrho(a) \leq m_0$. By (6.6) and assumption (i), $bm - b\varrho(b) - 1 = -bw'(b) > 0$; or $m - \varrho(b) > 1/b$. Therefore,

$$0 < 1/b < m - \varrho(b) \leq m - \varrho(a) \leq 1/A$$

by (5.5). Put $x := 1/(m - \varrho(b))$ and $y := 1/(m - \varrho(a))$. Then

$$a < A \leq y \leq x < b.$$

We write

$$\begin{aligned} aw'(a) &= -(m - \varrho(a))a + 1 = -(1/y)a + 1 > 0; \\ bw'(b) &= -(m - \varrho(b))b + 1 = -(1/x)b + 1 < 0. \end{aligned}$$

Making use of assumption (ii),

$$\int_{\{w=1\}} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} = \frac{1}{-(1/y)a + 1} - \frac{1}{-(1/x)b + 1} = \frac{by - ax}{(y - a)(b - x)}.$$

By Lemma 6.10 (i) then (ii),

$$\int_{\{w=1\}} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} \geq \frac{(b - a)y}{(y - a)(b - y)} \geq \frac{(b - a)A}{(A - a)(b - A)} = 2 \frac{a + b}{b - a} = 2 \coth(\mu((a, b))/2).$$

(b) Condition (b) together with (6.9) entails that $0 \leq m_0 - m + \varrho(a)$ and $0 \leq -m_0 + m - \varrho(b)$ whence $0 \leq \varrho(a) - \varrho(b)$ upon adding; so ϱ is constant on the interval $[a, b]$ by monotonicity. Define x and y as above. Then $x = y$ and $y \geq A$. The result now follows in a similar way to case (a).

(c) In this case,

$$\frac{1}{av'(a)} - \frac{1}{bv'(b)} \geq \frac{1}{av'_0(a)} - \frac{1}{bv'_0(b)} = 2 \coth(\mu((a, b))/2)$$

by Lemma 6.9.

(d) Condition (d) together with (6.9) means that $m_0 - m + \varrho(b) \leq 0$; that is, $m \geq \varrho(b) + m_0$. On the other hand, by Theorem 5.3, $m \leq \varrho(b) + m_0$. In consequence, $m = \varrho(b) + m_0$. It then follows that $\varrho \equiv 0$ on $[a, b]$ by Theorem 5.3. Now use Lemma 6.9. \square

Lemma 6.12. *Let $\phi : (0, +\infty) \rightarrow (0, +\infty)$ be a convex non-increasing function with $\inf_{(0, +\infty)} \phi > 0$. Let Λ be an at most countably infinite index set and $(x_h)_{h \in \Lambda}$ a sequence of points in $(0, +\infty)$ with $\sum_{h \in \Lambda} x_h < +\infty$. Then*

$$\sum_{h \in \Lambda} \phi(x_h) \geq \phi\left(\sum_{h \in \Lambda} x_h\right)$$

and the left-hand side takes the value $+\infty$ in case Λ is countably infinite and is otherwise finite.

Proof. Suppose $0 < x_1 < x_2 < +\infty$. By convexity $\phi(x_1) + \phi(x_2) \geq 2\phi(\frac{x_1+x_2}{2}) \geq \phi(x_1 + x_2)$ as ϕ is non-increasing. The result for finite Λ follows by induction. \square

Theorem 6.13. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be monotone increasing. Let (w, λ) solve (6.6). Assume that $w > 1$ on (a, b) . Then*

(i) for \mathcal{L}^1 -a.e. $t \in (1, \|w\|_\infty)$,

$$-\mu'_w \geq (2/t) \coth((1/2)\mu_w); \tag{6.10}$$

(ii) if $\varrho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (1, \|w\|_\infty)$ such that strict inequality holds in (6.10) for \mathcal{L}^1 -a.e. $t \in (1, t_0)$.

Proof. (i) Let $t \in (1, \|w\|_\infty)$ and assume that $\{w = t\} \cap Z = \emptyset$. We write $\{w > t\} = \bigcup_{h \in \Lambda} I_h$ where Λ is an at most countably infinite index set and $(I_h)_{h \in \Lambda}$ are disjoint non-empty well-separated open intervals in (a, b) . The term well-separated means that for each $h \in \Lambda$, $\inf_{k \in \Lambda \setminus \{h\}} d(I_h, I_k) > 0$. It follows from the fact that $w' \neq 0$ on ∂I_h for each $h \in \Lambda$. Put $\tilde{w} := w/t$ on $\{w > t\}$ so

$$\tilde{w}' + (mt)\tilde{w}^2 = (1/x + \varrho)\tilde{w} \text{ on } \{w > t\} \text{ and } \tilde{w} = 1 \text{ on } \{w = t\}.$$

We use the fact that the mapping $\phi : (0, +\infty) \rightarrow (0, +\infty); t \mapsto \coth t$ satisfies the hypotheses of Lemma 6.12. By Lemmas 6.11 and 6.12,

$$\begin{aligned} (0, +\infty] \ni \int_{\{w=t\}} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} &= (1/t) \int_{\{\tilde{w}=1\}} \frac{1}{|\tilde{w}'|} \frac{d\mathcal{H}^0}{\tau} \\ &= (1/t) \sum_{h \in \Lambda} \int_{\partial I_h} \frac{1}{|\tilde{w}'|} \frac{d\mathcal{H}^0}{\tau} \\ &\geq (2/t) \sum_{h \in \Lambda} \coth((1/2)\mu(I_h)) \\ &\geq (2/t) \coth((1/2) \sum_{h \in \Lambda} \mu(I_h)) \\ &= (2/t) \coth((1/2)\mu(\{w > t\})) = (2/t) \coth((1/2)\mu_w(t)). \end{aligned}$$

The statement now follows from Lemma 6.1.

(ii) Suppose that $\varrho \not\equiv 0$ on $[a, b]$. Put $\alpha := \inf\{\varrho > 0\} \in [a, b]$. Now that $\{w > t\} \uparrow (a, b)$ as $t \downarrow 1$ as $w > 1$ on (a, b) . Choose $t_0 \in (1, \|w\|_\infty)$ such that $\{w > t_0\} \cap (\alpha, b) \neq \emptyset$. Then for each $t \in (1, t_0)$ there exists $h \in \Lambda$ such that $\varrho \not\equiv 0$ on I_h . The statement then follows by Lemma 6.11. \square

Lemma 6.14. Let $\emptyset \neq S \subset \mathbb{R}$ be bounded and suppose S has the property that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. Then S is \mathcal{L}^1 -measurable and $\mathcal{L}^1(S) > 0$.

Proof. For each $s \in S$ put $t_s := \inf\{t > s : t \notin S\}$. Then $s < t_s < +\infty$, $[s, t_s) \subset S$ and $t_s \notin S$. Define

$$\mathcal{C} := \left\{ [s, t] : s \in S \text{ and } t \in (s, t_s) \right\}.$$

Then \mathcal{C} is a Vitali cover of S (see [6] Chapter 16 for example). By Vitali's Covering Theorem (cf. [6] Theorem 16.27) there exists an at most countably infinite subset $\Lambda \subset \mathcal{C}$ consisting of pairwise disjoint intervals such that

$$\mathcal{L}^1(S \setminus \bigcup_{I \in \Lambda} I) = 0.$$

Note that $I \subset S$ for each $I \in \Lambda$. Consequently, $S = \bigcup_{I \in \Lambda} I \cup N$ where N is an \mathcal{L}^1 -null set and hence S is \mathcal{L}^1 -measurable. The positivity assertion is clear. \square

Theorem 6.15. Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Let (w, λ) solve (6.6). Assume that $w > 1$ on (a, b) . Put $T := \min\{\|w_0\|_\infty, \|w\|_\infty\} > 1$. Then

- (i) $\mu_w(t) \leq \mu_{w_0}(t)$ for each $t \in [1, T]$;
- (ii) $\|w\|_\infty \leq \|w_0\|_\infty$;
- (iii) if $\varrho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (1, \|w\|_\infty)$ such that $\mu_w(t) < \mu_{w_0}(t)$ for each $t \in (1, t_0)$.

Proof. We adapt the proof of [15] Theorem I.6.1. The assumption entails that $\mu_w(1) = \mu_{w_0}(1) = \mu((a, b))$. Suppose for a contradiction that $\mu_w(t) > \mu_{w_0}(t)$ for some $t \in (1, T)$.

For $\varepsilon > 0$ consider the initial value problem

$$z' = \omega(t, z) + \varepsilon \text{ and } z(1) = \mu((a, b)) + \varepsilon \quad (6.11)$$

on $(0, T)$. Choose $v \in (0, 1)$ and $\tau \in (t, T)$. By [15] Lemma I.3.1 there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon < \varepsilon_0$ (6.11) has a solution z_ε defined on $[v, \tau]$ and this solution is unique by [15] Theorem I.3.1. Moreover, the sequence $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ converges uniformly to z_0 on $[v, \tau]$.

Given $0 < \varepsilon < \eta < \varepsilon_0$ it holds that $z_0 \leq z_\varepsilon \leq z_\eta$ on $[1, \tau]$ by [15] Theorem I.6.1. Note for example that $z'_0 \leq \omega(\cdot, z_0) + \varepsilon$ on $(1, \tau)$. In fact, $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ decreases strictly to z_0 on $(1, \tau)$. For if, say, $z_0(s) = z_\varepsilon(s)$ for some $s \in (1, \tau)$ then $z'_\varepsilon(s) = \omega(s, z_\varepsilon(s)) + \varepsilon > \omega(s, z_0(s)) = z'_0(s)$ by (6.11); while on the other hand $z'_\varepsilon(s) \leq z'_0(s)$ by considering the left-derivative at s and using the fact that $z_\varepsilon \geq z_0$ on $[1, \tau]$. This contradicts the strict inequality.

Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $z_\varepsilon(t) < \mu_w(t)$ for each $0 < \varepsilon < \varepsilon_1$. Now μ_w is right-continuous and strictly decreasing as $\mu_w(t) - \mu_w(s) = -\mu(\{s < w \leq t\}) < 0$ for $1 \leq s < t < \|w\|_\infty$ by continuity of w . So the set $\{z_\varepsilon < \mu_w\} \cap (1, t)$ is open and non-empty in $(0, +\infty)$ for each $\varepsilon \in (0, \varepsilon_1)$. Thus there exists a unique $s_\varepsilon \in [1, t)$ such that

$$\mu_w > z_\varepsilon \text{ on } (s_\varepsilon, t] \text{ and } \mu_w(s_\varepsilon) = z_\varepsilon(s_\varepsilon)$$

for each $\varepsilon \in (0, \varepsilon_1)$. As $z_\varepsilon(1) > \mu((a, b))$ it holds that each $s_\varepsilon > 1$. Note that $1 < s_\varepsilon < s_\eta$ whenever $0 < \varepsilon < \eta$ as $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ decreases strictly to z_0 as $\varepsilon \downarrow 0$.

Define

$$S := \{s_\varepsilon : 0 < \varepsilon < \varepsilon_1\} \subset (1, t).$$

We claim that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. This entails that S is \mathcal{L}^1 -measurable with positive \mathcal{L}^1 -measure by Lemma 6.14.

Suppose $s = s_\varepsilon \in S$ for some $\varepsilon \in (0, \varepsilon_1)$ and put $z := z_\varepsilon(s) = \mu_w(s)$. Put $k := \text{sech}^2(z_0(t))$. For $0 \leq \zeta < \eta < \varepsilon_1$ define

$$\Omega_{\zeta, \eta} := \{(u, y) \in \mathbb{R}^2 : u \in (0, t) \text{ and } z_\zeta(u) < y < z_\eta(u)\}$$

and note that this is an open set in \mathbb{R}^2 . We remark that for each $(u, y) \in \Omega_{\zeta, \eta}$ there exists a unique $\nu \in (\zeta, \eta)$ such that $y = z_\nu(u)$. Given $r > 0$ with $s + r < t$ set

$$Q = Q_r := \{(u, y) \in \mathbb{R}^2 : s \leq u < s + r \text{ and } |y - z| < \|z_\varepsilon - z\|_{C([s, s+r])}\}.$$

Choose $r \in (0, t - s)$ and $\varepsilon_2 \in (\varepsilon, \varepsilon_1)$ such that

- (a) $Q_r \subset \Omega_{0, \varepsilon_1}$;
- (b) $\|z_\varepsilon - z\|_{C([s, s+r])} < s\varepsilon/(2k)$;
- (c) $\sup_{\eta \in (\varepsilon, \varepsilon_2)} \|z_\eta - z\|_{C([s, s+r])} \leq \|z_\varepsilon - z\|_{C([s, s+r])}$;
- (d) $z_\eta < \mu_w$ on $[s + r, t]$ for each $\eta \in (\varepsilon, \varepsilon_2)$.

We can find $\delta \in (0, r)$ such that $z_\varepsilon < \mu_w < z_{\varepsilon_2}$ on $(s, s + \delta)$ as $z_{\varepsilon_2}(s) > z$; in other words, the graph of μ_w restricted to $(s, s + \delta)$ is contained in $\Omega_{\varepsilon, \varepsilon_2}$.

Let $u \in (s, s + \delta)$. Then $\mu_w(u) = z_\eta(u)$ for some $\eta \in (\varepsilon, \varepsilon_2)$ as above. We claim that $u = s_\eta$ so that $u \in S$. This implies in turn that $[s, s + \delta) \subset S$. Suppose for a contradiction that $z_\eta \not\prec \mu_w$ on

$(u, t]$. Then there exists $v \in (u, t]$ such that $\mu_w(v) = z_\eta(v)$. In view of condition (d), $v \in (u, s+r)$. By [1] Theorem 3.28 and Theorem 6.13,

$$\begin{aligned}\mu_w(v) - \mu_w(u) &= D\mu_w((u, v]) = D\mu_w^a((u, v]) + D\mu_w^s((u, v]) \\ &\leq D\mu_w^a((u, v]) = \int_u^v \mu'_w d\tau \leq \int_u^v \omega(\cdot, \mu_w) d\tau.\end{aligned}$$

On the other hand,

$$z_\eta(v) - z_\eta(u) = \int_u^v z'_\eta d\tau = \int_u^v \omega(\cdot, z_\eta) d\tau + \eta(v - u).$$

We derive that

$$\varepsilon(v - u) \leq \eta(v - u) \leq \int_u^v \left\{ \omega(\cdot, \mu_w) - \omega(\cdot, z_\eta) \right\} d\tau \leq k \int_u^v |\mu_w - z_\eta| d\mu$$

using the estimate (6.7). Thus

$$\begin{aligned}\varepsilon &\leq k \frac{1}{v - u} \int_u^v |\mu_w - z_\eta| d\mu \\ &\leq (k/s) \|\mu_w - z_\eta\|_{C([u, v])} \\ &\leq (k/s) \left\{ \|\mu_w - z\|_{C([s, s+r])} + \|z_\eta - z\|_{C([s, s+r])} \right\} \\ &\leq (2k/s) \|z_\varepsilon - z\|_{C([s, s+r])} < \varepsilon\end{aligned}$$

by (b) and (c) giving rise to the desired contradiction.

By Theorem 6.13, $\mu'_w \leq \omega(\cdot, \mu_w)$ for \mathcal{L}^1 -a.e. $t \in S$. Choose $s \in S$ such that μ_w is differentiable at s and the latter inequality holds at s . Let $\varepsilon \in (0, \varepsilon_1)$ such that $s = s_\varepsilon$. For any $u \in (s, t)$,

$$\mu_w(u) - \mu_w(s) > z_\varepsilon(u) - z_\varepsilon(s).$$

We deduce that $\mu'_w(s) \geq z'_\varepsilon(s)$. But then

$$\mu'_w(s) \geq z'_\varepsilon(s) = \omega(s, z_\varepsilon(s)) + \varepsilon > \omega(s, \mu_w(s)).$$

This strict inequality holds on a set of full measure in S . This contradicts Theorem 6.13.

(ii) Use the fact that $\|w\|_\infty = \inf\{t > 0 : \mu_w(t) > 0\}$.

(iii) Assume that $\varrho \not\equiv 0$ on $[a, b]$. Let $t_0 \in (1, \|w\|_\infty)$ be as in Lemma 6.13. Then for $t \in (1, t_0)$,

$$\begin{aligned}\mu_w(t) - \mu_w(1) &= D\mu_w((1, t]) = D\mu_w^a((1, t]) + D\mu_w^s((1, t]) \leq D\mu_w^a((1, t]) \\ &= \int_{(1, t]} \mu'_w ds < \int_{(1, t]} \omega(s, \mu_w) ds \leq \int_{(1, t]} \omega(s, \mu_{w_0}) ds = \mu_{w_0}(t) - \mu_{w_0}(1)\end{aligned}$$

by Theorem 6.13, Lemma 6.9 and the inequality in (i). □

Corollary 6.16. *Let $0 < a < b < +\infty$ and $0 \leq \varrho \in C([a, b])$ be non-decreasing. Suppose that (w, λ) solves (6.6). Assume that $w > 1$ on (a, b) . Let $0 \leq \varphi \in C^1((1, +\infty))$ be a strictly decreasing function such that $\int_{(1, +\infty)} |\varphi'| z_0 ds < +\infty$. Then*

$$(i) \int_a^b \varphi(w) d\mu \geq \int_a^b \varphi(w_0) d\mu;$$

(ii) equality holds in (i) if and only if $\varrho \equiv 0$ on $[a, b]$.

In particular,

$$(iii) \int_a^b \frac{1}{\sqrt{w^2 - 1}} d\mu \geq \pi \text{ with equality if and only if } \varrho \equiv 0 \text{ on } [a, b].$$

Proof. (i) Let $0 \leq \varphi \in C^1((1, +\infty))$ be a non-increasing function. By Tonelli's Theorem,

$$\begin{aligned} \int_{[1, +\infty)} \varphi' \mu_w ds &= \int_{[1, +\infty)} \varphi' \left\{ \int_{(a, b)} \chi_{\{w > s\}} d\mu \right\} ds \\ &= \int_{(a, b)} \left\{ \int_{[1, +\infty)} \varphi' \chi_{\{w > s\}} ds \right\} d\mu \\ &= \int_{(a, b)} \left\{ \varphi(w) - \varphi(1) \right\} d\mu = \int_{(a, b)} \varphi(w) d\mu - \varphi(1) \mu((a, b)) \end{aligned}$$

and a similar identity holds for μ_{w_0} . By Theorem 6.15, $\int_a^b \varphi(w) d\mu \geq \int_a^b \varphi(w_0) d\mu$.

Part (ii) follows from Theorem 6.15. (iii) flows from (i) and (ii) noting that the function $\varphi : (1, +\infty) \rightarrow \mathbb{R}; t \mapsto 1/\sqrt{t^2 - 1}$ satisfies the integral condition by Lemma 6.9. \square

The case $a = 0$. Let $0 < b < +\infty$ and $0 \leq \varrho \in C([0, b])$ be non-decreasing. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \varrho)u + \lambda = 0 \text{ on } (0, b) \text{ with } u(0) = 0 \text{ and } u(b) = 1 \quad (6.12)$$

where $u \in C([0, b]) \cap C^1((0, b))$ and $\lambda \in \mathbb{R}$. If $\varrho \equiv 0$ on $[0, b]$ then we write u_0 instead of u .

Lemma 6.17. *Let $0 < b < +\infty$ and $0 \leq \varrho \in C([0, b])$ be non-decreasing. Then*

- (i) *there exists a solution (u, λ) of (6.12) with $u \in C([0, b]) \cap C^1((0, b))$ and $\lambda \in \mathbb{R}$;*
- (ii) *λ is given by $\lambda = -g(b)/G(b)$ where $G := \int_0^b g ds$;*
- (iii) *the pair (u, λ) in (i) is unique;*
- (iv) *$u > 0$ on $(0, b]$.*

Proof. (i) The function $u : [a, b] \rightarrow \mathbb{R}$ given by

$$u = \frac{g(b)}{G(b)} \frac{G}{g} \quad (6.13)$$

on $[0, b]$ solves (6.12) with λ as in (ii). (iii) Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (6.12). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \varrho)u + \lambda = 0 \text{ on } (0, b) \text{ with } u(0) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda G + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. (iv) follows from the formula (6.13) and unicity. \square

Lemma 6.18. *Suppose $-\infty < a < b < +\infty$ and that $\phi : [a, b] \rightarrow \mathbb{R}$ is convex with $\phi \in C^1((a, b))$. Suppose that there exists $\xi \in (a, b)$ such that*

$$\phi(\xi) = \frac{b - \xi}{b - a} \phi(a) + \frac{\xi - a}{b - a} \phi(b).$$

Then

$$\phi(c) = \frac{b - c}{b - a} \phi(a) + \frac{c - a}{b - a} \phi(b)$$

for each $c \in [a, b]$.

Proof. Let $c \in (\xi, b)$. By monotonicity of chords,

$$\frac{\phi(\xi) - \phi(a)}{\xi - a} \leq \frac{\phi(c) - \phi(\xi)}{c - \xi}$$

so

$$\begin{aligned} \phi(c) &\geq \frac{c-a}{\xi-a}\phi(\xi) - \frac{c-\xi}{\xi-a}\phi(a) \\ &= \frac{c-a}{\xi-a} \left\{ \frac{b-\xi}{b-a}\phi(a) + \frac{\xi-a}{b-a}\phi(b) \right\} - \frac{c-\xi}{\xi-a}\phi(a) \\ &= \frac{b-c}{b-a}\phi(a) + \frac{c-a}{b-a}\phi(b) \end{aligned}$$

and equality follows. The case $c \in (a, \xi)$ is similar. \square

Lemma 6.19. *Let $0 < b < +\infty$ and $0 \leq \varrho \in C([0, b])$ be non-decreasing. Let (u, λ) satisfy (6.12). Then*

- (i) $u \geq u_0$ on $[0, b]$;
- (ii) if $\varrho \not\equiv 0$ on $[0, b]$ then $u > u_0$ on $(0, b)$.

Proof. (i) The mapping $G : [0, b] \rightarrow [0, G(b)]$ is a bijection with inverse G^{-1} . Define $\eta : [0, G(b)] \rightarrow \mathbb{R}$ via $\eta := (tg) \circ G^{-1}$. Then

$$\eta' = \frac{(tg)'}{g} \circ G^{-1} = (2 + t\varrho) \circ G^{-1}$$

on $(0, G(b))$ so η' is non-decreasing there. This means that η is convex on $[0, G(b)]$. In particular, $\eta(s) \leq [\eta(G(b))/G(b)]s$ for each $s \in [0, G(b)]$. For $t \in [0, b]$ put $s := G(t)$ to obtain $tg(t) \leq (bg(b)/G(b))G(t)$. A rearrangement gives $u \geq u_0$ on $[0, b]$ noting that $u_0 : [0, b] \rightarrow \mathbb{R}; t \mapsto t/b$. (ii) Suppose $\varrho \not\equiv 0$ on $[0, b]$. Suppose that $u(c) = u_0(c)$ for some $c \in (0, b)$. Then $\eta(G(c)) = [\eta(G(b))/G(b)]G(c)$. By Lemma 6.18, η' is constant on $(0, G(b))$. This implies that $\varrho \equiv 0$ on $[0, b]$. \square

Lemma 6.20. *Let $0 < b < +\infty$. Then $\int_0^b \frac{u_0}{\sqrt{1-u_0^2}} d\mu = \pi/2$.*

Proof. The integral is elementary as $u_0(t) = t/b$ for $t \in [0, b]$. \square

7 Proof of Main Theorem

Lemma 7.1. *Let $x \in H$ and v be a unit vector in \mathbb{R}^2 such that the pair $\{x, v\}$ forms a positively oriented orthogonal basis for \mathbb{R}^2 . Put $b := (\tau, 0)$ where $|x| = \tau$ and $\gamma := \theta(x) \in (0, \pi)$. Let $\alpha \in (0, \pi/2)$ such that*

$$\frac{\langle v, x - b \rangle}{|x - b|} = \cos \alpha.$$

Then

- (i) $C(x, v, \alpha) \cap H \cap \overline{C}(0, e_1, \gamma) = \emptyset$;
- (ii) for any $y \in C(x, v, \alpha) \cap H \setminus \overline{B}(0, \tau)$ the line segment $[b, y]$ intersects \mathbb{S}_τ^1 outside the closed cone $\overline{C}(0, e_1, \gamma)$.

We point out that $C(0, e_1, \gamma)$ is the open cone with vertex 0 and axis e_1 which contains the point x on its boundary. We note that $\cos \alpha \in (0, 1)$ because

$$\langle v, x - b \rangle = -\langle v, b \rangle = -\langle (1/\tau)Ox, b \rangle = -\langle Op, e_1 \rangle = \langle x, O^*e_1 \rangle = \langle x, e_2 \rangle > 0 \quad (7.1)$$

and if $|x - b| = \langle v, x - b \rangle$ then $b = x - \lambda v$ for some $\lambda \in \mathbb{R}$ and hence $x_1 = \langle e_1, x \rangle = \tau$ and $x_2 = 0$.

Proof. (i) For $\omega \in \mathbb{S}^1$ define the open half-space

$$H_\omega := \{y \in \mathbb{R}^2 : \langle y, \omega \rangle > 0\}.$$

We claim that $C(x, v, \alpha) \subset H_v$. For given $y \in C(x, v, \alpha)$,

$$\langle y, v \rangle = \langle y - x, v \rangle > |y - x| \cos \alpha > 0.$$

On the other hand, it holds that $\overline{C}(0, e_1, \gamma) \cap H \subset \overline{H}_{-v}$. This establishes (i).

(ii) By some trigonometry $\gamma = 2\alpha$. Suppose that ω is a unit vector in $C(b, -e_1, \pi/2 - \alpha)$. Then $\lambda := \langle \omega, e_1 \rangle < \cos \alpha$ since upon rewriting the membership condition for $C(b, -e_1, \pi/2 - \alpha)$ we obtain the quadratic inequality

$$\lambda^2 - 2\cos^2 \alpha \lambda + \cos \gamma > 0.$$

For ω a unit vector in $\overline{C}(0, e_1, \gamma)$ the opposite inequality $\langle \omega, e_1 \rangle \geq \cos \alpha$ holds. This shows that

$$C(b, -e_1, \pi/2 - \alpha) \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}_\tau^1 = \emptyset.$$

The set $C(x, v, \alpha) \cap H$ is contained in the open convex cone $C(b, -e_1, \pi/2 - \alpha)$. Suppose $y \in C(x, v, \alpha) \cap H \setminus \overline{B}(0, \tau)$. Then the line segment $[b, y]$ is contained in $C(b, -e_1, \pi/2 - \alpha) \cup \{b\}$. Now the set $C(b, -e_1, \pi/2 - \alpha) \cap \mathbb{S}_\tau^1$ disconnects $C(b, -e_1, \pi/2 - \alpha) \cup \{b\}$. This entails that $(b, y] \cap C(b, -e_1, \pi/2 - \alpha) \cap \mathbb{S}_\tau^1 \neq \emptyset$. The foregoing paragraph entails that $(b, y] \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}_\tau^1 = \emptyset$. This establishes the result. \square

Lemma 7.2. *Let E be an open set in \mathbb{R}^2 such that $M := \partial E$ is a C^2 hypersurface in \mathbb{R}^2 . Assume that $E \setminus \{0\} = E^{sc}$. Suppose*

$$(i) \ x \in (M \setminus \{0\}) \cap H;$$

$$(ii) \ \sin(\sigma(x)) = -1.$$

Then E is not convex.

Proof. Let $\gamma_1 : I \rightarrow M$ be a C^2 parametrisation of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. As $\sin(\sigma(x)) = -1$, $n(x)$ and hence $n_1(0)$ point in the direction of x . Put $v := -t_1(0) = -t(x)$. We may write

$$\gamma_1(s) = \gamma_1(0) + st_1(0) + R_1(s) = x - sv + r_1(s)$$

for $s \in I$ where $R_1(s) = s \int_0^1 \gamma_1'(ts) - \gamma_1'(0) dt$ and we can find a finite positive constant K such that $|R_1(s)| \leq Ks^2$ on a symmetric open interval I_0 about 0 with $I_0 \subset \subset I$. Then

$$\frac{\langle \gamma_1(s) - x, v \rangle}{|\gamma_1(s) - x|} = \frac{\langle -sv + R_1, v \rangle}{|-sv + R_1|} = \frac{1 - \langle (R_1/s), v \rangle}{|v - R_1/s|} \rightarrow 1$$

as $s \uparrow 0$. Let α be as in Lemma 7.1 with x and v as just mentioned. The above estimate entails that $\gamma_1(s) \in C(x, v, \alpha)$ for small $s < 0$. By (4.6) and Lemma 4.4 the function r_1 is non-increasing on I . In particular, $r_1(s) \geq r_1(0) = |x| =: \tau$ for $I \ni s < 0$ and $\gamma_1(s) \notin B(0, \tau)$.

Choose $\delta_1 > 0$ such that $\gamma_1(s) \in C(x, v, \alpha) \cap H$ for each $s \in [-\delta_1, 0)$. Put $\alpha := \inf\{s \in [-\delta_1, 0] : r_1(s) = 0\}$. Suppose first that $\alpha \in [-\delta_1, 0)$. Then E is not convex (see Lemma 4.2). Now suppose that $\alpha = 0$. Let γ be as in Lemma 7.1. Then the open circular arc $\mathbb{S}_\tau^1 \setminus \overline{C}(0, e_1, \gamma)$ does not intersect \overline{E} as $E \setminus \{0\} = E^{sc}$ (for otherwise M fails to be C^1 at x by Lemma 4.2). Choose $s \in [-\delta_1, 0)$. Then the points b and $\gamma_1(s)$ lie in \overline{E} . But by Lemma 7.1 the line segment $[b, \gamma_1(s)]$ intersects \mathbb{S}_τ^1 in $\mathbb{S}_\tau^1 \setminus \overline{C}(0, e_1, \gamma)$. Let $c \in [b, \gamma_1(s)] \cap \mathbb{S}_\tau^1$. Then $c \notin \overline{E}$. This shows that \overline{E} is not convex. But if E is convex then \overline{E} is convex. Therefore E is not convex. \square

Theorem 7.3. *Let f be as in (1.3). Given $v > 0$ let E be a minimiser of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 , $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. Put*

$$R := \inf\{\varrho > 0\} \in [0, +\infty). \quad (7.2)$$

Then $\Omega \cap (R, +\infty) = \emptyset$ with Ω as in (4.9).

Proof. Suppose that $\Omega \cap (R, +\infty) \neq \emptyset$. As Ω is open in $(0, +\infty)$ by Lemma 4.6 we may write Ω as a countable union of disjoint open intervals in $(0, +\infty)$. By a suitable choice of one of these intervals we may assume that $\Omega = (a, b)$ for some $0 \leq a < b < +\infty$ and that $\Omega \cap (R, +\infty) \neq \emptyset$. Let us assume for the time being that $a > 0$. Note that $[a, b] \subset \pi(M)$ and $\cos \sigma$ vanishes on $M_a \cup M_b$.

Let $y : \Omega \rightarrow [-1, 1]$ be as in (4.11). Then y has a continuous extension to $[a, b]$ and $y = \pm 1$ at $\tau = a, b$. This may be seen as follows. For $\tau \in (a, b)$ the set $M_\tau \cap \overline{H}$ consists of a singleton by Lemma 4.4. The limit $x := \lim_{\tau \downarrow a} M_\tau \cap \overline{H} \in \mathbb{S}_a^1 \cap \overline{H}$ exists as M is C^1 . There exists a C^2 parametrisation $\gamma_1 : I \rightarrow M$ with $\gamma_1(0) = x$ as above. By (4.6) and Lemma 4.4, r_1 is decreasing on I . So $r_1 > a$ on $I \cap \{s < 0\}$ for otherwise the C^1 property fails at x . It follows that $\gamma_1 = \gamma \circ r_1$ and $\sigma_1 = \sigma \circ \gamma \circ r_1$ on $I \cap \{s < 0\}$. Thus $\sin(\sigma \circ \gamma) \circ r_1 = \sin \sigma_1$ on $I \cap \{s < 0\}$. Now the function $\sin \sigma_1$ is continuous on I . So $y \rightarrow \sin \sigma_1(0) \in \{\pm 1\}$ as $\tau \downarrow a$. Put $\eta_1 := y(a)$ and $\eta_2 := y(b)$.

Let us consider the case $\eta = (\eta_1, \eta_2) = (1, 1)$. According to Theorem 3.6 the generalised mean curvature $H_f(\cdot, E)$ is constant on $M \setminus \{0\}$ with value λ (say). Put $u := y$. Note that $u < 1$ on (a, b) for otherwise $\cos(\sigma \circ \gamma)$ vanishes at some point in (a, b) bearing in mind Lemma 4.4. By Theorem 4.7 the pair (u, λ) satisfies (6.4) with $\eta = (1, 1)$. By Lemma 6.2, $u > 0$ on $[a, b]$. Put $w := 1/u$. Then $(w, -\lambda)$ satisfies (6.6) and $w > 1$ on (a, b) . By Lemma 4.8,

$$\theta_2(b) - \theta_2(a) = \int_a^b \theta'_2 d\tau = - \int_a^b \frac{u}{\sqrt{1-u^2}} \frac{d\tau}{\tau} = - \int_a^b \frac{1}{\sqrt{w^2-1}} \frac{d\tau}{\tau}.$$

By Corollary 6.16, $|\theta_2(b) - \theta_2(a)| > \pi$. But this contradicts the definition of θ_2 in (4.10) as θ_2 takes values in $(0, \pi)$ on (a, b) . A similar argument deals with the case $\eta = (-1, -1)$.

Now let us consider the case $\eta = (1, -1)$. As above,

$$\theta_2(b) - \theta_2(a) = - \int_a^b \frac{u}{\sqrt{1-u^2}} \frac{d\tau}{\tau}.$$

By Corollary 6.7, $\theta_2(b) - \theta_2(a) > 0$. This means that $\theta_2(b) \in (0, \pi]$. As before the limit $x := \lim_{\tau \uparrow b} M_\tau \cap \overline{H} \in \mathbb{S}_b^1 \cap \overline{H}$ exists as M is C^1 . Using a local parametrisation it can be seen that $\theta_2(b) = \theta(x)$ and $\sin(\sigma(x)) = -1$. If $\theta_2(b) \in (0, \pi)$ then E is not convex by Lemma 7.2. This contradicts Theorem 4.10. Suppose then that $\theta_2(b) = \pi$. Then the points $(-b, 0)$ and $(a, 0)$ belong to \overline{E} . But any point of the form point $(-z, 0)$ for $a < z < b$ does not. So again \overline{E} and hence E fails to be convex. A similar argument works for the case $\eta = (-1, 1)$.

Suppose finally that $a = 0$. By Lemma 4.5, $u(0) = 0$ and $u(b) = \pm 1$. Suppose $u(b) = 1$. As above,

$$\theta_2(b) - \theta_2(0) = - \int_a^b \frac{u}{\sqrt{1-u^2}} \frac{d\tau}{\tau} < - \int_a^b \frac{u_0}{\sqrt{1-u_0^2}} \frac{d\tau}{\tau} = -\pi/2$$

by Lemma 6.19, the fact that the function $\varphi : (0, 1) \rightarrow \mathbb{R}; t \mapsto t/\sqrt{1-t^2}$ is strictly increasing and Lemma 6.20. This means that $\theta_2(0) > \pi/2$. This contradicts the C^1 property at $0 \in M$. The case $u(b) = -1$ is similar. \square

Lemma 7.4. *Let f be as in (1.3). Let $v > 0$. Then there exists a minimiser E of (1.2) such that ∂E consists of a countable union of disjoint centred circles whose radii accumulate at 0 if at all.*

Proof. Let E be a minimiser of (1.2) such that E is open, $M := \partial E$ is a C^1 hypersurface in \mathbb{R}^2 , $M \setminus \{0\}$ is a C^2 hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. First observe that

$$\begin{aligned} \emptyset \neq \pi(M \setminus \{0\}) &= \left[\pi(M \setminus \{0\}) \cap (0, R] \right] \cup \left[\pi(M \setminus \{0\}) \cap (R, +\infty) \right] \\ &= \left[\pi(M \setminus \{0\}) \cap (0, R] \right] \cup \left[\pi(M \setminus \{0\}) \cap (R, +\infty) \right] \setminus \Omega \end{aligned}$$

by Lemma 7.3. We assume that the latter member is non-empty. By definition of Ω , $\cos \sigma = 0$ on $M \cap A((R, +\infty))$. Let $\tau \in \pi(M \setminus \{0\}) \cap (R, +\infty)$. We claim that $M_\tau = \mathbb{S}_\tau^1$. Suppose for a contradiction that $M_\tau \neq \mathbb{S}_\tau^1$. By Lemma 4.2, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 . Let x be a point on the boundary of one of these spherical arcs relative to \mathbb{S}_τ^1 . There exists a C^2 parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $\gamma_1(I) \subset A((R, +\infty))$ as $\tau > R$. By (4.6), $r'_1 = 0$ on I as $\cos \sigma_1 = 0$ on I because $\cos \sigma = 0$ on $M \cap A((R, +\infty))$; that is, r_1 is constant on I . This means that $\gamma_1(I) \subset \mathbb{S}_\tau^1$. As the function $\sin \sigma_1$ is continuous on I it takes the value ± 1 there. By (4.7), $r_1 \theta'_1 = \sin \sigma_1$ on I . This means that θ_1 is either strictly decreasing or strictly increasing on I . This entails that the point x is not a boundary point of M_τ in \mathbb{S}_τ^1 .

Assume that $R > 0$. It follows from these considerations that $M \setminus \overline{B}(0, R)$ consists of a finite union of disjoint centred circles. Note that $f \geq e^{h(0)} =: c > 0$ on \mathbb{R}^2 . As a result, $+\infty > P_f(E) \geq cP(E)$ and in particular the relative perimeter $P(E, \mathbb{R}^2 \setminus \overline{B}(0, R)) < +\infty$. This explains why $M \setminus \overline{B}(0, R)$ comprises only finitely many circles.

We claim that only one of the possibilities

$$M_R = \emptyset, M_R = \mathbb{S}_R^1, M_R = \{Re_1\} \text{ or } M_R = \{-Re_1\}$$

holds. To prove this suppose that $M_R \neq \emptyset$ and $M_R \neq \mathbb{S}_R^1$. Bearing in mind Lemma 4.2 we may choose $x \in M_R$ such that x lies on the boundary of M_R relative to \mathbb{S}_R^1 . Assume that $x \in H$. Let $\gamma_1 : I \rightarrow M$ be a local parametrisation of M with $\gamma_1(0) = x$ with the usual conventions. We first notice that $\cos(\sigma(x)) = 0$ for otherwise we obtain a contradiction to Theorem 7.3. As r_1 is decreasing on I and x is a relative boundary point it holds that $r_1 < \tau$ on $I \cap \{s > 0\}$. Put $I^+ := I \cap \{s > 0\}$. According to Theorem 3.6 and (3.8) the curvature $H(\cdot, E) = k$ of $\gamma_1(I^+) \cap B(0, r)$ is constant. Hence $\gamma_1(I^+) \cap B(0, r)$ consists of a line or circular arc. The fact that $\cos(\sigma(x)) = 0$ means that $\gamma_1(I^+) \cap B(0, r)$ cannot be a line. So $\gamma_1(I^+) \cap B(0, r)$ is an open arc of a circle C containing x in its closure with centre on the line segment $[0, x]$ and radius $0 < r < \tau$. Because M is closed and C^1 with constant curvature $1/r$ it holds that $C \subset M$. But this contradicts the fact that $E \setminus \{0\} = E^{sc}$.

Suppose that $M_R = \emptyset$. As both sets M and \mathbb{S}_R^1 are compact, $d(M, \mathbb{S}_R^1) > 0$. Assume first that $\mathbb{S}_R^1 \subset E$. Put $F := B(0, R) \setminus E$ and suppose $F \neq \emptyset$. Then F is a set of finite perimeter, $F \subset\subset B(0, R)$ and $P(F) = P(E, B(0, R))$. Let B be a centred ball with $|B| = |F|$. By the classical isoperimetric inequality, $P(B) \leq P(F)$. Define $E_1 := (\mathbb{R}^2 \setminus B) \cap (B(0, R) \cup E)$. Then $V_f(E_1) = V_f(E)$ and $P_f(E_1) \leq P_f(E)$. Now suppose that $\mathbb{S}_R^1 \subset \mathbb{R}^2 \setminus \overline{E}$. In like fashion we may redefine E via $E_1 := B \cup (E \setminus \overline{B}(0, R))$ with B a centred ball in $B(0, R)$. The remaining cases above can be dealt with in a similar vein. The upshot of this argument is that M consists of a finite union of disjoint centred circles in case $R > 0$.

If $R = 0$ on the other hand then M consists of a countable union of disjoint centred circles whose radii accumulate at 0 if at all. \square

Lemma 7.5. *Suppose that the function $J : [0, +\infty) \rightarrow [0, +\infty)$ is continuous non-decreasing and $J(0) = 0$. Let $N \in \mathbb{N} \cup \{+\infty\}$ and $\{t_h : h = 0, \dots, 2N+1\}$ a sequence of points in $[0, +\infty)$ with*

$$t_0 > t_1 > \dots > t_{2h} > t_{2h+1} > \dots \geq 0.$$

Then

$$+\infty \geq \sum_{h=0}^{2N+1} J(t_h) \geq J\left(\sum_{h=0}^{2N+1} (-1)^h t_h\right).$$

Proof. We suppose that $N = +\infty$. The series $\sum_{h=0}^{\infty} (-1)^h t_h$ converges by the alternating series test. For each $n \in \mathbb{N}$,

$$\sum_{h=0}^{2n+1} (-1)^h t_h \leq t_0$$

and the same inequality holds for the infinite sum. As in Step 2 in [4] Theorem 2.1,

$$+\infty \geq \sum_{h=0}^{\infty} J(t_h) \geq J(t_0) \geq J\left(\sum_{h=0}^{\infty} (-1)^h t_h\right)$$

as J is non-decreasing. \square

Proof of Theorem 1.1. There exists a minimiser E of (1.2) with the property that ∂E consists of a countable union of disjoint centred circles whose radii accumulate at 0 if at all according to Lemma 7.4. As such we may write

$$E = \bigcup_{h=0}^N A((a_{2h+1}, a_{2h}))$$

where $N \in \mathbb{N} \cup \{+\infty\}$ and $+\infty > a_0 > a_1 > \dots > a_{2h} > a_{2h+1} > \dots \geq 0$. Define

$$\begin{aligned} \mathbf{f} &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto e^{h(t)}; \\ g &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto t\mathbf{f}(t); \\ G &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto \int_0^t g \, d\tau. \end{aligned}$$

Then $G : [0, +\infty) \rightarrow [0, +\infty)$ is a bijection with inverse G^{-1} . Define the non-decreasing function

$$J : [0, +\infty) \rightarrow \mathbb{R}; t \mapsto g \circ G^{-1}.$$

With $\{a_0, a_1, \dots\}$ as above put $t_h := G(a_h)$ for $h = 0, \dots, 2N+1$. Then $+\infty > t_0 > t_1 > \dots > t_{2h} > t_{2h+1} > \dots \geq 0$. Put $B := B(0, r)$ where $r := G^{-1}(v/2\pi)$ so that $V_f(B) = v$. Note that

$$v = V_f(E) = 2\pi \sum_{h=0}^N \left\{ G(a_{2h}) - G(a_{2h+1}) \right\} = 2\pi \sum_{h=0}^{2N+1} (-1)^h t_h.$$

By Lemma 7.5,

$$P_f(E) = 2\pi \sum_{h=0}^{2N+1} g(a_h) = 2\pi \sum_{h=0}^{2N+1} J(t_h) \geq 2\pi J\left(\sum_{h=0}^{2N+1} (-1)^h t_h\right) = 2\pi J(v/2\pi) = P_f(B).$$

\square

Proof of Corollary 1.2. We may assume that h diverges to infinity; for otherwise, the density f is constant on \mathbb{R}^2 and the result follows by the classical isoperimetric inequality (cf. [8], [11]). By Theorem 2.1, the problem (1.2) admits a bounded minimiser E .

Extend h to \mathbb{R} via

$$\tilde{h} := \begin{cases} h & \text{on } [0, +\infty); \\ h(0) & \text{on } (-\infty, 0); \end{cases}$$

then \tilde{h} is convex and non-decreasing on \mathbb{R} . Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers (see e.g. [1] 2.1) with support in $[-\varepsilon, \varepsilon]$ and set $\tilde{h}_\varepsilon := \tilde{h} \star \psi_\varepsilon$ on \mathbb{R} for each $\varepsilon > 0$; put $h_\varepsilon := \tilde{h}_\varepsilon|_{[0, +\infty)}$. Then h_ε

is a non-decreasing convex function on $[0, +\infty)$ and $h_\varepsilon \in C^\infty((0, +\infty))$ for each $\varepsilon > 0$; moreover, $(h_\varepsilon)_{\varepsilon>0}$ converges to h locally uniformly on $[0, +\infty)$ as $\varepsilon \downarrow 0$.

Let B be a centred ball in \mathbb{R}^2 with $V_f(E) = V_f(B)$. For each $k \in \mathbb{N}$ let us write h_k instead of $h_{1/k}$ for the sake of legibility and f_k for the corresponding density. Denote by B_k the open centred ball in \mathbb{R}^2 with volume $V_{f_k}(B_k) = V_{f_k}(E)$ for each $k \in \mathbb{N}$. Then $V_{f_k}(E) \rightarrow V_f(E)$ as $k \rightarrow \infty$ as (f_k) converges to f locally uniformly on \mathbb{R}^2 and consequently $V_{f_k}(B_k) \rightarrow V_f(B)$ as $k \rightarrow \infty$. Suppose that the centred ball B_k has radius r_k and note that $r_k > 0$. Likewise, denote by $r > 0$ the radius of B . Define $\mathbf{F}_k := 2\pi \int_0^\tau t e^{h_k} dt$ on $[0, +\infty)$ for each $k \in \mathbb{N}$ and likewise for \mathbf{F} . As $h_k \geq h(\cdot - 1/k)$ on $[1/k, +\infty)$,

$$\begin{aligned} \mathbf{F}_k(\tau) &= 2\pi \int_0^\tau t e^{h_k} dt \geq 2\pi \int_{1/k}^\tau t e^{h_k} dt \geq 2\pi \int_{1/k}^\tau t e^{h(t-1/k)} dt \\ &\geq 2\pi \int_{1/k}^\tau (t - 1/k) e^{h(t-1/k)} dt = \mathbf{F}(\tau - 1/k) \end{aligned}$$

for $\tau \geq 1/k$. The above inequality entails that the sequence (r_k) is bounded as $\mathbf{F}_k(r_k) = V_{f_k}(B_k)$ for each $k \in \mathbb{N}$. By extracting a subsequence if necessary we may assume that the sequence (r_k) converges to some $r_0 \in [0, +\infty)$ as $k \rightarrow +\infty$. It follows that

$$\mathbf{F}_k(r_k) - \mathbf{F}(r_0) = \mathbf{F}_k(r_k) - \mathbf{F}(r_k) + \mathbf{F}(r_k) - \mathbf{F}(r_0) \rightarrow 0$$

as $k \rightarrow +\infty$ because (\mathbf{F}_k) converges locally uniformly to \mathbf{F} on $[0, +\infty)$ and \mathbf{F} is continuous. On the other hand, $\mathbf{F}_k(r_k) = V_{f_k}(B_k) \rightarrow V_f(B) = \mathbf{F}(r)$ as $k \rightarrow +\infty$. Thus $\mathbf{F}(r_0) = \mathbf{F}(r)$ and $r_0 = r$ as the function \mathbf{F} is strictly increasing. It then follows that

$$P_{f_k}(B_k) = 2\pi r_k e^{h_k(r_k)} \rightarrow 2\pi r e^{h(r)} = P_f(B)$$

as $k \rightarrow \infty$.

By Theorem 1.1,

$$P_{f_k}(B_k) \leq P_{f_k}(E)$$

for each $k \in \mathbb{N}$. On taking limits we obtain

$$P_f(B) \leq P_f(E) \text{ and } V_f(B) = V_f(E).$$

This shows that B is a minimiser for (1.2). □

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